Evaluating Discrete Dynamic Strategies in Affine Models

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Abstract

We consider the problem of measuring the performance of a dynamic strategy, re-balanced at a discrete set of dates, whose objective is that of hedging a claim in an incomplete market driven by a general multi-dimensional affine process. The main purpose of the paper is to propose a method to efficiently compute the expected value and variance of the hedging error of the strategy. Representing the payoff of the claim as an inverse Laplace transform, we are able to get semi-explicit formulas for strategies satisfying a certain property. The result is quite general and can be applied to a very rich class of models and strategies, including Delta hedging. We provide illustrations for the case of Heston stochastic volatility model.

1 Introduction

One of the most discussed assumptions of financial models, especially criticized in periods of financial turmoils, is that of market completeness, that is the perfect replication of any contingent claim by a suitable dynamic trading strategy. Theoretically, this is often achieved by ruling out any market imperfection, like illiquidity, credit risks, transactions costs, taxes, etc and by assuming the possibility of continuous time trading. Of course, real markets usually fail to satisfy most, if not all of such assumptions. One of the main challenges for financial economics is therefore to address such issues, by proposing models with less stringent hypotheses or by studying what happens when they do not hold. In this paper we focus on the impossibility of trading continuously in time. Even if all other assumptions of the model are satisfied, the inherent discreteness of trading times is a source of market incompleteness in the real world. The aim of the paper is to efficiently evaluate the impact of trading in discrete time on the final goal of the strategy in a framework that incorporates other sources of incompleteness like jumps and stochastic volatility.

The object of our investigation is the ex-ante assessment of the performances of dynamic trading strategies. Probably, the most notable instance of such problem is measuring the hedging error of a strategy, based on a liquid assets, that tries to hedge a future liability. Problems of such kind arise when hedging either a claim using futures contracts, or a payoff of a derivative security with a Delta hedging strategy based on the underlying asset, and in any case when a dynamic strategy is adopted. Ex-ante, a possible way to measure the performance of a strategy is by evaluating expected value and variance of its hedging error. This is usually done by approximations (see Angelini and Herzel (2009) and references therein) or by Monte Carlo simulations. The approach proposed in the paper, based on Laplace transforms, is a generalization of Angelini and Herzel (2009) and allows to efficiently perform such computations for a very wide class of models.

Our methodology can be applied to many important models for financial markets for equities, interest rates and credit products where the stochastic dynamics of risk factors are driven by affine processes. Affine processes are popular in modeling financial markets, because of their analytical tractability and flexibility. Their defining property is that their conditional characteristic function is computable as an exponentially affine function of the state variables at current time. Affine processes in finance were first introduced for interest rate models by Vasicek (1977), Cox, Ingersoll and Ross (1985) and by Duffie and Kan (1996) who gave a general formulation in a multivariate setting. The pioneering work in the case of equities is due to Heston (1993) who proposed a stochastic volatility model and opened the way to many other models of the kind. Duffie, Filipović and Schachermayer (2003) formulated a general theory for affine processes for equities and interest rates. Further application to credit risk modeling can be found in Duffie and Singleton (2003). We present the general model framework of our approach in Section 2, where we provide the necessary details of affine processes as well as some examples. One of the reasons for exploring alternative ways for efficient computations in the setting of Affine processes is that the use of Monte Carlo simulations is often not so simple. For example, in the case of Heston model, it is well known that discretizing the dynamics of the process may introduce serious numerical problems, especially when the non-absorbing condition on the volatility process is not satisfied, which is often the case when the model is calibrated on market data, and hence more sophisticated and computationally intensive methods are called for, like for instance that proposed by Broadie and Kaja (2006), which requires numerical inversions of a characteristic function. Even when Monte Carlo is a viable alternative, like for instance in the CIR model, semi-closed formulas may be a better choice when one has to solve optimization problems, or computing derivatives with respect to some parameters of the model. We show some examples of such issues in the Section devoted to the application.

The problem of measuring the hedging error in discrete time was first addressed by Toft (1996) who proposed an approximation for the variance of the Delta hedging strategy in the Black-Scholes model. Hayashi and Mykland (2005) use a weak convergence argument to derive the asymptotic distribution of the hedging error as the number of trades goes to infinity. Their approach was generalized by Tankov and Voltchkova (2009) to Lévy processes with jumps. A very important problem, related to this, is that of determining a strategy that minimizes the variance of the hedging error. An extremely rich branch of the financial literature flourished after the seminal paper of Föllmer and Sondermann (1986). Schweizer (1995) contains a review of the main results and contributions in a discrete time setting. In continuous time, Černý and Kallsen (2008) solve the problem of computing the optimal strategy and the optimal variance for the Heston model, and Kallsen and Vierthauer (2009) extended the results to general affine stochastic volatility models.

An important ingredient of our method is that of representing the payoff of the claim as an inverse Laplace transform. This approach, introduced to mathematical finance by Carr and Madan (1999), was proposed in the context of variance-optimal hedging strategies for Lévy processes by Černý (2007) and Hubalek et al. (2006). From this, the key idea, proposed by Angelini and Herzel (2009) still in the case of Lévy processes, is to express the Delta-based strategies as an inverse Laplace transform, so that one can directly compute the Laplace transform of the hedging error and hence its expected value and variance. The present paper extends the approach to the more general setting of Affine processes, addressing more interesting problems related to hedging under stochastic volatility.

We show how to apply the transform approach to a class of dynamic strategies that can be represented as integrals of affine exponentials satisfying some integrability conditions. The fulfillment of such conditions is related to some results on the existence of exponential moments of affine processes such as those in Glasserman and Kim (2010) or in Andersen and Piterbarg (2007), which will be explored in Section 4, with a detailed analysis of the Heston model. In this case, we will also provide an example of the rather counter-intuitive, but not inexplicable, result that the hedging strategy based on the Delta that is correct according to the model performs worse, in the presence of correlation between volatility and returns, than the simple Black-Scholes Delta, which ignores both correlation and stochastic volatility.

2 The general framework

Let us consider a contingent claim with a payoff H at time T that can be expressed as

$$H = \int_{\mathcal{C}} e^{zy_T} \Pi(dz), \tag{2.1}$$

where $\mathcal{C} = R + i\mathbb{R}$, with $R \in \mathbb{R}$, Π is a finite complex measure on $\mathcal{C}$ and $y_T = \ln(S_T)$, where S is the price of a risky asset. In other words, the payoff function is represented as an inverse Laplace transform. For instance, the payoff at time T of a European call option on S_T with strike price $K > 0$ may be written as

$$(e^y - K)^+ = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} e^{zy} \frac{K^{1-z}}{z(z-1)} dz,$$

for an arbitrary $R > 1$. Other examples are the put, the power call and the digital option, see Hubalek et al. (2006). Although in this approach $\mathcal{C} \subset \mathbb{C}$ may be more general, we prefer to stick with a vertical line for the sake of a clearer exposition.

We assume that the dynamics of the market variables are driven by a multi-dimensional affine process X, whose components include $y = \ln(S)$ and possibly other factors as stochastic volatility, dividend yields, etc. A relevant example is the model proposed by Heston (1993), where X is a two-dimensional process of the logarithm of the asset price and its instantaneous variance. We will examine this case in Section 4. Pan (2002) studied a four-dimensional affine model combining stochastic volatility, interest rates and dividend yield. However, for the sake of a clearer exposition, we do not treat here the case of stochastic interest rates, and we suppose that the risk-free rate is zero. A very general study of affine processes in a financial setting is contained in Duffie, Pan and Singleton (2000). Another important area of application is to model the intensity of defaults in evaluating the credit risk (Duffie and Singleton (2003)). Duffie, Filipović and Schachermayer (2003) provide a characterization of affine processes.

As the theory of affine processes is well established, we only recall those concepts that are necessary for our purpose, referring to Duffie, Filipović and Schachermayer (2003) for a more complete exposition and technical details.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t < \infty}, P)$ be a filtered probability space. We interpret P as the physical or objective probability measure [1]. We consider a time-homogeneous Markov process X defined in a state space $D \subset \mathbb{R}^d$ and its conditional characteristic function [2]

$$\phi(u, X_t, t, s) \quad = \quad E_t\left[e^{u \cdot X_s}\right], \tag{2.2}$$

where $u \in i\mathbb{R}^d$, $t, s \in [0, T]$ with $t \leq s$, E_t denotes the expected value conditional on $\mathcal{F}_t$ and $\cdot$ the scalar product. When X is affine, its characteristic function is analytically computable as

$$\phi(u, X_t, t, s) \quad = \quad e^{\alpha(u,t,s)+\beta(u,t,s)\cdot X_t}, \tag{2.3}$$

[1] We mention that an approach present in the literature is to compute hedging errors under the risk-neutral probability, in other words when the model is calibrated on market prices, see e.g. Denkl et al. (2011) and the references therein.

[2] This is sometimes called the *moment generating function*, here we adopt the terminology used by Duffie, Filipović and Schachermayer (2003) or Filipović (2009).

where $\alpha(u,t,s)$ and $\beta(u,t,s)$ are functions from $i\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+$ to $\mathbb{C}$ and to $\mathbb{C}^d$ respectively, satisfying a system of Riccati equations whose general form is given in Theorem 2.7 in Duffie, Filipović and Schachermayer (2003) Such equations have an explicit solution in some particular cases, otherwise can be numerically integrated. For example, in the Black-Scholes model, where X is a one-dimensional Brownian motion with coefficients μ and σ

$$\alpha^{bs}(u,t,s) = \left((\mu - \frac{\sigma^2}{2})u + \frac{\sigma^2}{2}u^2\right)(s-t), \tag{2.4}$$

$$\beta^{bs}(u,t,s) = u. \tag{2.5}$$

For our application it is necessary to compute the joint characteristic function of X at times $t \le t_1 \le \cdots \le t_\nu \le T$ conditional on the information in t,

$$\phi_\nu(u_1,\ldots,u_\nu,X_t,t,t_1,\ldots,t_\nu) = E_t\left[e^{\sum_{j=1}^{\nu} u_j\cdot X_{t_j}}\right] \tag{2.6}$$

where $u_k \in i\mathbb{R}^d$ for $k = 1,\ldots,\nu$. For an affine process we have

$$\phi_\nu(u_1,\ldots,u_\nu,X_t,t,t_1,\ldots,t_\nu) = e^{\alpha_\nu(u_1,\ldots,u_\nu,t,t_1,\ldots,t_\nu)+\beta_\nu(u_1,\ldots,u_\nu,t,t_1,\ldots,t_\nu)\cdot X_t}, \tag{2.7}$$

where the functions $\alpha_\nu(\cdot)$ and $\beta_\nu(\cdot)$ are equal to $\alpha(\cdot)$ and $\beta(\cdot)$ for $\nu = 1$ and can be computed recursively if $\nu > 1$,

$$\begin{aligned}\phi_\nu(u_1,\ldots,u_\nu,X_t,t,t_1,\ldots,t_\nu) &= E_t\left[e^{u_1\cdot X_{t_1}}E_{t_1}\left[e^{\sum_{j=2}^{\nu} u_j\cdot X_{t_j}}\right]\right]\\ &= E_t\left[e^{u_1\cdot X_{t_1}}\phi_{\nu-1}(u_2,\ldots,u_\nu,X_{t_1},t_1,t_2,\ldots,t_\nu)\right]\end{aligned}$$

Therefore,

$$\begin{aligned}\alpha_\nu(u_1,\ldots,u_\nu,t,t_1,\ldots,t_\nu) &= \alpha_{\nu-1}(u_2,\ldots,u_\nu,t_1,t_2,\ldots,t_\nu)\\ &+ \alpha(u_1+\beta_{\nu-1}(u_2,\ldots,u_\nu,t_1,t_2,\ldots,t_\nu),t,t_1)\\ \beta_\nu(u_1,\ldots,u_\nu,t,t_1,\ldots,t_\nu) &= \beta(u_1+\beta_{\nu-1}(u_2,\ldots,u_\nu,t_1,t_2,\ldots,t_\nu),t,t_1)\end{aligned}$$

where the recursions are always well defined because of Propositions 6.1 and 6.4 of Duffie, Filipović and Schachermayer (2003).

We suppose that the functions $\alpha(u,t,T)$ and $\beta(u,t,T)$ can be analytically extended to an open convex domain U containing $0 \in \mathbb{C}^d$ for all $t \in [0,T]$, and we assume that

$$2R1_y \in U \cap \mathbb{R}^d \tag{2.8}$$

$$21_y \in U \cap \mathbb{R}^d \tag{2.9}$$

where 1_y denotes the d-dimensional vector of zeros except for the entry corresponding to y that is equal to one. Assumption (2.9) implies that S_t is square-integrable and Assumption (2.8) that

$$E[e^{2Ry_T}] < \infty \tag{2.10}$$

It follows that the contingent claim H is square integrable:

$$E[|H|^2] \le E\left[|\Pi|(\mathcal{C}) \int_{\mathcal{C}} e^{2Ry_T} |\Pi|(dz)\right] = (|\Pi|(\mathcal{C}))^2 \, E[e^{2Ry_T}] < \infty$$

where $|\Pi|$ denotes the total variation measure of Π in the sense of Rudin (1987) (Section 6.1).

3 Dynamic hedging strategies

In this section we give the main definitions and properties of dynamic hedging strategies in our setting and we measure the hedging error of a class of strategies in terms of expected value and variance.

We consider the problem of hedging the contingent claim H when trading is only allowed at a finite and prefixed set of dates from time 0 until maturity T, $0 = t_0 < t_1 < \ldots < t_N = T$.

Let $\vartheta = (\vartheta_{t_k})$, for $k = 0, \ldots, N-1$, be a stochastic process representing a trading strategy. The random variable ϑ_{t_k} is the number of shares of S held from time t_k up to time t_{k+1}. We assume that it depends only on the information available at time t_k, i.e. that it is $\mathcal{F}_{t_k}$-measurable.

The final value of strategy ϑ starting from an initial capital c is

$$G_T(\vartheta) = c + \sum_{k=1}^{N} \vartheta_{t_{k-1}} \Delta S_k \tag{3.11}$$

where

$$\Delta S_k = S_{t_k} - S_{t_{k-1}}$$

and its hedging error is given by

$$\varepsilon(\vartheta, c) = H - G_T(\vartheta). \tag{3.12}$$

Definition 3.1 *A trading strategy ϑ has an affine representation if*

$$\vartheta_{t_k} = \int_{\mathcal{C}} e^{a(z,t_k)+b(z,t_k)\cdot X_{t_k}} \Pi(dz), \tag{3.13}$$

for all $k = 0, \dots, N-1$, where $a(z,t_k)$ and $b(z,t_k)$ are functions from $\mathbb{C}\times\mathbb{R}_+$ to $\mathbb{C}$ and to $\mathbb{C}^d$ respectively, such that the functions

$$f(z,t_k) = e^{2Re(a(z,t_k))} \tag{3.14}$$

$$g(z,t_k,t_j) = E\left[e^{\left(2Re(b(z,t_k))+\beta(21_y,t_k,t_j)\right)\cdot X_{t_k}}\right] \tag{3.15}$$

are bounded on $\mathcal{C}$, for $j = k, k+1$, where $Re(v)$ denotes the real part of vector $v \in \mathbb{C}^d$.

The assumption on functions f and g in (3.14) and (3.15) are technical conditions that allow us to use Fubini's Theorem in the proof of the main result (Theorem 3.1). They must be checked case by case, depending on the model and on the strategy. These conditions could also be formulated in terms of the domain U, but one should distinguish the positive components of X from the arbitrary ones as it is illustrated in Section 4.1 for various strategies in the case of the stochastic volatility Heston model. They imply that the hedging error (3.12) is square-integrable. Indeed

$$E\left[|\theta_{t_k} S_{t_j}|^2\right] \le E\left[|\Pi|(\mathcal{C}) \int_{\mathcal{C}} |e^{a(z,t_k)+b(z,t_k)\cdot X_{t_k}} S_{t_j}|^2 |\Pi|(dz)\right] \tag{3.16}$$

Interchanging the integrals by Fubini's Theorem and, for $j = k+1$, using the law of iterated expectations and Assumption (2.9), we get

$$\begin{aligned}
&\int_{\mathcal{C}} E\left[|e^{a(z,t_k)+b(z,t_k)\cdot X_{t_k}} S_{t_j}|^2\right] |\Pi|(dz) = \\
&\int_{\mathcal{C}} e^{2\mathrm{Re}(a(z,t_k))+\alpha(21_y,t_k,t_j)} \times \\
&E\left[e^{\left(2\mathrm{Re}(b(z,t_k))+\beta(21_y,t_k,t_j)\right)\cdot X_{t_k}}\right] |\Pi|(dz) < \infty
\end{aligned} \tag{3.17}$$

In principle the contour $\mathcal{C}$ and the measure Π appearing in the definition of affine strategies may be different from those related to the claim H in Formula (2.1). This may be the case when the hedging strategy is performed by aiming at a different claim, or we may also study trading strategies that are not intended to hedge any claim. However, to fix ideas, we will consider only hedging strategies for the claim H.

For an example of a trading strategy with affine representation let us assume that X is also affine under a pricing measure Q,

$$E_t^Q\left[e^{u\cdot X_s}\right] = e^{\bar{\alpha}(u,t,s)+\bar{\beta}(u,t,s)\cdot X_t} \tag{3.18}$$

where the functions $\bar{\alpha}(u,t,s)$ and $\bar{\beta}(u,t,s)$ solve a system of Riccati equations depending on the risk-neutral dynamics of X. Conditions for a process to be affine under both measures P and Q are given by Duffie, Pan and Singleton (2000). From (3.18) and using (the conditional version of) Fubini's Theorem, we get an expression for the value at time t of a European claim with payoff expressed as in (2.1)

$$\begin{aligned} H_t &= E_t^Q[H] \\ &= E_t^Q\left[\int_{\mathcal{C}} e^{zy_T}\Pi(dz)\right] \\ &= \int_{\mathcal{C}} E_t^Q\left[e^{z1_y\cdot X_T}\right]\Pi(dz) \\ &= \int_{\mathcal{C}} e^{\bar{\alpha}(z1_y,t,T)+\bar{\beta}(z1_y,t,T)\cdot X_t}\Pi(dz). \end{aligned} \tag{3.19}$$

To use Fubini's Theorem here we suppose that $E_t^Q[e^{Ry_T}] < \infty$. For the last equality, we suppose that $z1_y$ belongs to the domain of analytical extension of $\bar{\alpha}(u,t,T)$ and $\bar{\beta}(u,t,T)$, for all $z \in \mathcal{C}$.

By differentiating (3.19), we can compute the sensitivities of the pricing formula with respect to the factors of the model. In particular, the Delta of the claim at time t is given by

$$\begin{aligned} \Delta_t^H &= \frac{\partial H_t}{\partial S_t} \\ &= e^{-y_t}\frac{\partial}{\partial y_t}\int_{\mathcal{C}} e^{\bar{\alpha}(z1_y,t,T)+\bar{\beta}(z1_y,t,T)\cdot X_t}\Pi(dz) \\ &= \int_{\mathcal{C}} \bar{\beta}(z1_y,t,T)\cdot 1_y e^{\bar{\alpha}(z1_y,t,T)+(\bar{\beta}(z1_y,t,T)-1_y)\cdot X_t}\Pi(dz) \end{aligned} \tag{3.20}$$

provided that the derivative with respect to y_t exists and that integration and differentiation can be interchanged. Hence, the Delta strategy has an affine representation with

$$\begin{aligned} a(z,t) &= \ln\left(\bar{\beta}(z1_y,t,T)\cdot 1_y\right) + \bar{\alpha}(z1_y,t,T) \\ b(z,t) &= \bar{\beta}(z1_y,t,T) - 1_y. \end{aligned}$$

The functions $\bar{\alpha}(\cdot)$ and $\bar{\beta}(\cdot)$ depend on the pricing model. In particular, for the Black-Scholes model (where $1_y = 1$) they are given by (2.4) and (2.5), with $\mu = 0$. Analogously, differentiating with respect to other state variables gives strategies which have an affine representation.

A common strategy, often used in practice for hedging with futures contracts, is that obtained by regressing the value of a less liquid security H to a more liquid instrument S and for this reason is usually called *Beta strategy*. More precisely, the shares held at time t_k are

$$B_{t_k} = \frac{\mathrm{cov}_{t_k}\left(H_{t_{k+1}}, S_{t_{k+1}}\right)}{\mathrm{var}_{t_k}[S_{t_{k+1}}]} \tag{3.21}$$

In our setting this may be computed as

$$\begin{aligned} B_{t_k} &= \frac{E_{t_k}\left[H_{t_{k+1}}S_{t_{k+1}}\right] - E_{t_k}\left[H_{t_{k+1}}\right]E_{t_k}\left[S_{t_{k+1}}\right]}{E_{t_k}\left[S^2_{t_{k+1}}\right] - E_{t_k}\left[S_{t_{k+1}}\right]^2} \\ &= \frac{1}{\phi(2\ 1_y, X_{t_k}, t_k, t_{k+1}) - \phi(1_y, X_{t_k}, t_k, t_{k+1})^2} \times \\ &\quad \int_{\mathcal{C}} \Big(E_{t_k}\left[e^{\bar{\alpha}(z1_y,t_{k+1},T)+\left(\bar{\beta}(z1_y,t_{k+1},T)+1_y\right)\cdot X_{t_{k+1}}}\right] + \\ &\quad -E_{t_k}\left[e^{\bar{\alpha}(z1_y,t_{k+1},T)+\bar{\beta}(z1_y,t_{k+1},T)\cdot X_{t_{k+1}}}\right] E_{t_k}\left[e^{1_y\cdot X_{t_{k+1}}}\right]\Big)\,\Pi(dz) \\ &= \frac{1}{\phi(2\ 1_y, X_{t_k}, t_k, t_{k+1}) - \phi(1_y, X_{t_k}, t_k, t_{k+1})^2} \times \\ &\quad \int_{\mathcal{C}} e^{\bar{\alpha}(z1_y,t_{k+1},T)}\left(\phi(\bar{\beta}(z1_y,t_{k+1},T)+1_y, X_{t_k}, t_k, t_{k+1}) + \right. \\ &\quad \left. -\phi(\bar{\beta}(z1_y,t_{k+1},T), X_{t_k}, t_k, t_{k+1})\phi(1_y, X_{t_k}, t_k, t_{k+1})\right)\Pi(dz) \end{aligned}$$

We see that the Beta strategy may still be represented as an integral, but in general not with an affine integrand. However, it has an affine representation when X is a Lévy process (see Theorem 2.1 in Hubalek et al. (2006)).

The Beta strategy, being a one-step regression, has a similar structure as the so called locally risk-minimizing strategy, that is the strategy minimizing the variance of costs over the next period, which is obtained by backward regressions (see Schweizer (1995) for a complete exposition on variance-optimal hedging in discrete time). For our purposes, the Beta strategy is relevant because, when $P = Q$ and in any case when the underlying is a martingale, the two coincide. Moreover, they are also globally optimal, i.e. they minimize the variance of the hedging error (3.12). Although the variance of the hedging error of the Beta strategy can not be computed from our result below, this strategy serves as a benchmark term for the various hedge ratios at the initial time in our illustration in Section 4.3. As far as we know, computing the variance of the Beta strategy or that of the optimal strategy for discrete time trading in affine processes is still an open problem. There are results when allowing continuous time portfolio rebalancing: Černý and Kallsen (2008) compute the globally optimal strategy and its variance in Heston model, while Kallsen and Pauwels (2009) solve the problem for general affine stochastic volatility models, in case the underlying is a martingale, and Kallsen and Vierthauer (2009) extend the approach of Černý and Kallsen (2008) to more general affine processes.

The hedging error (3.12) of a strategy with an affine representation for a contingent claim whose payoff can be written as (2.1), has the following integral representation

$$\varepsilon(\vartheta, c) = -c + \int_{\mathcal{C}} \left(e^{z y_T} - \sum_{k=1}^{N} e^{a(z,t_{k-1})+b(z,t_{k-1})\cdot X_{t_{k-1}}} \Delta S_k \right) \Pi(dz). \qquad (3.22)$$

Under the assumptions of Section 2, in particular Assumptions (2.8) and (2.9), the above representation and the analytical tractability of affine processes can be exploited to compute expected value and variance of the hedging error as shown in the following result.

Theorem 3.1 *Let H be a contingent claim satisfying condition (2.1), ϑ be strategy which has an affine representation (Definition 3.1), and c be the initial capital, then*

$$E[\varepsilon(\vartheta, c)] = \int_{\mathcal{C}} e(z)\Pi(dz) - c \qquad (3.23)$$

and

$$E[\varepsilon(\vartheta,0)^2] = \int_{\mathcal{C}}\int_{\mathcal{C}}(v_1(w,z) - v_2(w,z) - v_3(w,z) + v_4(w,z))\Pi(dw)\Pi(dz) \tag{3.24}$$

where

$$\begin{aligned} e(z) &= \phi(z1_y, X_0, 0, T) - \sum_{k=1}^{N} e^{a(z,t_{k-1})} \times \\ &\quad (\phi_2(b(z,t_{k-1}), 1_y, X_0, 0, t_{k-1}, t_k) - \\ &\quad \phi(b(z,t_{k-1}) + 1_y, X_0, 0, t_{k-1})) \end{aligned}$$

$$v_1(w,z) = \phi((w+z)1_y, X_0, 0, T)$$

$$\begin{aligned} v_2(w,z) &= \sum_{k=1}^{N} e^{a(w,t_{k-1})} \times \\ &\quad (\phi_3(b(w,t_{k-1}), 1_y, z1_y, X_0, 0, t_{k-1}, t_k, T) - \\ &\quad \phi_2(b(w,t_{k-1}) + 1_y, z1_y, X_0, 0, t_{k-1}, T)) \end{aligned}$$

$$v_3(w,z) = v_2(z,w).$$

To write $v_4(w,z)$*, we define*

$$m(j_1,j_2,k_1,k_2) = \\ \begin{cases} \phi_4(b(w,t_{j_1}), 1_y, b(z,t_{k_1}), 1_y, X_0, 0, t_{j_1}, t_{j_2}, t_{k_1}, t_{k_2}) \\ \text{for } j_1 \le j_2 \le k_1 \le k_2; \\ \phi_4((b(z,t_{k_1}), 1_y, b(w,t_{j_1}), 1_y, X_0, 0, t_{k_1}, t_{k_2}, t_{j_1}, t_{j_2}) \\ \text{for } k_1 \le k_2 \le j_1 \le j_2 \end{cases}$$

Then we have

$$\begin{aligned} v_4(w,z) &= \sum_{j=1}^{N}\sum_{k=1}^{N} e^{a(w,t_{j-1})} e^{a(z,t_{k-1})} \times \\ &\quad (m(j-1,j,k-1,k) \\ &\quad -m(j-1,j,k-1,k-1) \\ &\quad -m(j-1,j-1,k-1,k) \\ &\quad +m(j-1,j-1,k-1,k-1)) \,. \end{aligned}$$

Therefore, the variance of the hedging error is

$$\mathrm{var}(\varepsilon(\vartheta, c)) = \mathrm{var}(\varepsilon(\vartheta, 0)) = E[\varepsilon(\vartheta, 0)^2] - E[\varepsilon(\vartheta, 0)]^2.$$

Proof. See the Appendix.

Theorem 3.1 states that the expected value and the variance of the hedging error may be represented respectively as a one-dimensional and a two-dimensional inverse Laplace transforms. It may be used to study the effects of model mispecification or trader's personal views, in terms of hedging strategies and parameters, on the performance of the hedge. This because the claim, the model and the strategy are completely independent from each other. An interesting example of how to exploit this flexibility is the case of a given underlying model generating the data (say Heston model with a certain set of parameters) and a given strategy like the Black-Scholes Delta strategy or the model Delta strategy implemented using model parameters different from those of the data generating model.

Formulas (3.23) and (3.24) can be computed with a numerical inversion of one- and two-dimensional Laplace transforms. For more details on this as well as numerical integration schemes we refer to Angelini and Herzel (2009). For general affine processes, differently from the Lévy process setting there considered, numerical tests on algorithm precision are not so immediate. This is because even for a static, buy and hold strategy, that is when $N = 1$, in general there are not closed formulas available, like in the case of Lévy processes, to compute the expected value and the variance of the hedging error to serve as a comparison result (there are in this context too, like for the CIR model, but stochastic interest rates are not treated in the present paper). One straightforward way to proceed is through numerical integration. Moreover, Monte Carlo simulations are not totally trustable in terms of precision, as discretization schemes may give numerical problems, especially when certain condition on the (positive components of the) process are not satisfied, while more exact methods rely on the inversion of the characteristic function, as for instance shown in Broadie and Kaja (2006) for the Heston model. In any case, we will perform a test through simulations of the two-dimensional inversion algorithm in the case of Heston model at the end of Section 4.3. We remark that, even when Monte Carlo is a viable alternative, semi-closed formulas may be important to solve optimization problems, like that to find the Strategy 2.(b) of Section 4.2, or to compute the sensitiveness with respect to some parameters of the model by taking their derivative, as we will show in Section 4.3.

We observe that, when $N = 1$, an alternative to our formula is to compute the variance of the payoff and its covariance with the underlying at maturity. The variance of the underlying at maturity comes from the characteristic function of the driving process X. We write the expectations involved $E[H]$, $E[H^2]$ and $E[HS_T]$ using the representation as inverse Laplace transform of the payoffs of respectively the call, the power call and the self-quanto call (see Hubalek et al. (2006)) and Fubini's Theorem analogously to (3.19). Then we adopt the one-dimensional inversion algorithm. We will use such approach as a comparison term in the test at the end of Section 4.3.

4 The Heston Model

In this section we illustrate an important instance of the general framework presented in Section 2, that is the case of Heston model, and we apply Theorem 3.1 to compare the errors produced by different strategies to hedge a European call option. We start by studying the explosion of the exponential moments that is necessary to determine the domain of analytical extension of the characteristic function, then we examine some hedging strategies and show how to check if they are affine or not. We conclude by showing some applications of the methodology.

4.1 Conditional Characteristic Function

For a simpler exposition, we assume that the pricing measure and the objective measure are equal and that, as in the rest of the paper, the risk-free rate is zero. In this case, the dynamics of the process $X_t = (y_t, v_t)$ are

$$\begin{aligned} dy_t &= -\frac{1}{2} v_t dt + \sqrt{v_t} dW_t^1 \\ dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_t^2, \end{aligned} \tag{4.25}$$

where W^1 and W^2 are correlated Brownian motion, $d\langle W^1, W^2\rangle_t = \rho dt$, and $\sigma > 0$.

For convenience of the reader, we derive the conditional characteristic function (2.2) for this model, for $u = (u_1, u_2)$,

$$E_t\left[e^{u_1 y_T + u_2 v_T}\right] = e^{\alpha(u,t,T) + \beta_1(u,t,T) y_t + \beta_2(u,t,T) v_t}$$

For $\tau = T - t$, the Riccati equations (see Filipović (2009), Theorem 10.1, or Duffie, Pan and Singleton (2000), Equations (2.5) and (2.6)) for $\alpha(\tau) = \alpha(u, t, T)$ and $\beta_i(\tau) = \beta_i(u, t, T)$, $i = 1, 2$, are

$$\begin{aligned}
\dot{\alpha}(\tau) &= \kappa\theta\beta_2(\tau) \\
\dot{\beta}_1(\tau) &= 0 \\
\dot{\beta}_2(\tau) &= -\frac{1}{2}\beta_1(\tau) - \kappa\beta_2(\tau) + \frac{1}{2}\beta_1(\tau)^2 + \rho\sigma\beta_1(\tau)\beta_2(\tau) + \frac{1}{2}\sigma^2\beta_2(\tau)^2
\end{aligned}$$

with boundary conditions $\alpha(0) = 0$, $\beta_1(0) = u_1$ and $\beta_2(0) = u_2$, which immediately lead to $\beta_1(\tau) = u_1$ and to the following equations

$$\begin{aligned}
\dot{\alpha}(\tau) &= \kappa\theta\beta_2(\tau) \\
\dot{\beta}_2(\tau) &= \frac{1}{2}\sigma^2\beta_2(\tau)^2 + (\rho\sigma u_1 - \kappa)\beta_2(\tau) + \frac{1}{2}u_1(u_1 - 1)
\end{aligned} \tag{4.26}$$

The point now is to solve the second equation and then simply integrate the first equation. Setting

$$\begin{aligned}
B &= \rho\sigma u_1 - \kappa \\
D &= B^2 - \sigma^2 u_1(u_1 - 1) \\
d &= \sqrt{D} \\
a &= \frac{-B + d}{\sigma^2} \\
b &= \frac{-B - d}{\sigma^2}
\end{aligned}$$

where $\sqrt{\cdot}$ denotes the analytical extension of the real square root to $\mathbb{C} \setminus \mathbb{R}_-$, one finds

$$\begin{aligned}
\beta_1(\tau) &= u_1 \\
\beta_2(\tau) &= \frac{b(a - u_2) - a(b - u_2)e^{-d\tau}}{(a - u_2) - (b - u_2)e^{-d\tau}} \\
\alpha(\tau) &= \frac{\kappa\theta}{\sigma^2}\left(-2\log\left(\frac{(a - u_2) - (b - u_2)e^{-d\tau}}{a - b}\right) + (B - d)\tau\right)
\end{aligned} \tag{4.27}$$

In the expression for $\alpha(\tau)$, some care has to be taken when computing the complex logarithm, because of the well known problem of its different branches. However, the above form should avoid numerical problems, as reported by Lord and Kahl (2006), and also confirmed by our experience.

We are interested now in analyzing the set $M(\tau) \subset \mathbb{R}^d$ where the exponential moments of X_τ are finite

$$M(\tau) = \left\{u \in \mathbb{R}^d \mid E\left[e^{u \cdot X_\tau}\right] < \infty\right\} \tag{4.28}$$

In the case of diffusion processes, by Theorem 10.3 of Filipović (2009), the domain of analytical extension U of $\alpha(\tau)$ and $\beta(\tau)$ contains the strip

$$\mathcal{S}(M(\tau)) = \left\{u \in \mathbb{C}^2 \mid \mathrm{Re}(u) \in M(\tau)\right\}$$

and

$$U \cap \mathbb{R}^2 = M(\tau) \tag{4.29}$$

Moreover, $M(\tau)$ has the non-expanding property in τ, i.e. for all $0 \leq \tau_1 \leq \tau_2$, $M(\tau_1) \supseteqq M(\tau_2)$. To determine $M(\tau)$, we use the argument by Glasserman and Kim (2010) (Section 3) or by Andersen and Piterbarg (2007) (Proposition 3.1), based on the analysis of the differential equation (4.26). For each $u \in \mathbb{R}^2$, let us define the time of moment explosion

$$\tau^\star(u) = \sup\left\{t \geq 0 \mid E\left[e^{u \cdot X_t}\right] < \infty\right\}$$

so that

$$M(\tau) = \left\{u \in \mathbb{R}^2 \mid \tau < \tau^\star(u)\right\}$$

The result in Glasserman and Kim (2010) reads as follows:

- when $D > 0$, if $u_2 \leq a$ then $\tau^\star(u) = \infty$ and if $u_2 > a$ then

$$\tau^\star(u) = \frac{1}{d} \log \frac{b - u_2}{a - u_2}$$

- when $D = 0$, if $u_2 \leq a = b$, then $\tau^\star(u) = \infty$ and if $u_2 > a$ then[3]

$$\tau^\star(u) = \frac{2}{\sigma^2(u_2 - a)}$$

- when $D < 0$, then

$$\tau^\star(u) = \frac{1}{\sqrt{-D}}\left(\pi - 2\tan^{-1}\left(\frac{\sigma^2 u_2 + B}{\sqrt{-D}}\right)\right)$$

[3]In Glasserman and Kim (2010) there is a typo for this case.

Note that, given a fixed time to maturity τ, by inverting the result above on the time of moment explosion, one can directly compute the set $M(\tau)$. For each $u_1 \in \mathbb{R}$, we define the maximal value $u_2^\star \in \mathbb{R}$ such that $(u_1, u_2) \in M(\tau)$ for $u_2 < u_2^\star$, namely

$$u_2^\star = \sup\left\{u_2 \in \mathbb{R} \mid E\left[e^{u \cdot X_\tau}\right] < \infty\right\}$$

It can be explicitly computed as follows:

- If $D > 0$, then

$$u_2^\star = \frac{a - be^{-\sqrt{D}\tau}}{1 - e^{-\sqrt{D}\tau}}$$

- If $D = 0$, then

$$u_2^\star = a + \frac{2}{\sigma^2 \tau}$$

- If $D < 0$, when $u_1 \in (u_1^-, u_1^+)$ then

$$u_2^\star = \frac{\sqrt{-D}\tan(\frac{\pi - \sqrt{-D}\tau}{2}) - B}{\sigma^2}$$

  where u_1^- and u_1^+ are the two solutions of $\sqrt{-D}\tau = 2\pi$, else

$$u_2^\star = -\infty$$

The third point of the result means that, for $u_1 \geq u_1^+$ or $u_1 \leq u_1^-$, the exponential moment $E\left[e^{u \cdot X_\tau}\right]$ always explodes.

Figure 1 provides a graphical illustration of these results for parameters $\theta = 0.05$, $\kappa = 3$, $\sigma = 0.5$. The top panel shows the set $M(T)$ in the plane (u_1, u_2), when $\rho = 0.5$ for $T = 0.5$, $T = 1$ and $T = \infty$. The bottom panel shows $M(T)$ for different correlation coefficients $\rho = -0.9, 0, 0.9$, when $T = 1$. Note, in the top panel, the non expanding property in τ and the fact that vertical sections of $M(T)$ are half-lines and horizontal sections are bounded intervals. In the bottom panel we observe the symmetry between opposite values of the correlation. For positive ρ, the right tail of the log-returns is fatter than the left one, hence the moments $E\left[S_T^{u_1}\right]$ exist for smaller values of u_1 than in case of negative ρ. For $u_1 = 0$, the corresponding $u_2^\star$ obviously does not depend on ρ. It is also interesting to look at the joint moments $E\left[S_T^{u_1} e^{u_2 v_T}\right]$. In case of negative correlation between log-returns and their volatility, the intersection with the positive quadrant $u_1 \geq 0$ and $u_2 \geq 0$ is much wider than in the case of positive correlation, as it is expected.

4.2 Strategies

Let us now consider the following strategies:

1. Model Delta. Under our assumptions the price at time t of the contingent claim H is given by $H_t = E_t[H]$ and its dynamics are

$$dH_t = \Delta_t^H dS_t + \mathcal{V}_t^H dv_t + (\dots)dt$$

where Δ_t^H and $\mathcal{V}_t^H$ are the partial derivatives with respect to $S = \exp(y)$ and v. The model Delta is equal to Δ_t^H, and represents the strategy that hedges the risk of the underlying (but ignores the volatility risk). It is given by (3.20) and (4.27).

2. Black-Scholes Delta. This is the case of a trader who wishes to adopt the standard Black-Scholes strategy, neglecting some elements of the true dynamics of the underlying. The only parameter the trader has to choose is the volatility parameter σ_t^{bs} at time t. If σ_t^{bs} does not depend on y_t, the strategy is given by (3.20)

$$\Delta_t^{bs} = \int_{\mathcal{C}} e^{\ln(z)+\bar{\alpha}^{bs}(z,t,T)+(z-1)y_{t_k}}\Pi(dz), \tag{4.30}$$

with the function $\bar{\alpha}^{bs}(\cdot)$ of the Black-Scholes model as in (2.4), where μ is set to zero and $\sigma = \sigma_t^{bs}$. For some choices of σ_t^{bs} the strategy has an affine representation (leaving aside the technical assumptions (3.14) and (3.15) that will be checked later). This obviously holds when σ_t^{bs} is deterministic, with

$$\begin{aligned} a(z,t) &= \ln(z) + \bar{\alpha}^{bs}(z,t,T) \\ b(z,t) &= (z-1)1_y. \end{aligned}$$

Otherwise it has to be checked case by case. We consider three choices for the volatility parameter σ_t^{bs}, the first two are deterministic, the last one stochastic:

(a) The implied volatility of the option at time 0.

(b) The *optimal hedging volatility*, that is the volatility parameter that minimizes the variance of the error produced by the Black-Scholes Delta hedging strategy, i.e. the solution to the problem

$$\min_{\sigma\geq 0} \text{var}\left(H - \sum_{k=1}^{N} \Delta_{t_{k-1}}^{bs}(\sigma)\Delta S_k\right),$$

where $\Delta^{bs}_{t_{k-1}}(\sigma)$ is the Black-Scholes Delta at time t_{k-1} as a function of the volatility parameter σ. The optimal value can be determined numerically from Theorem 3.1 (the constant volatility is calculated so that the variance given by (3.24) using (4.30) as hedge ratio is minimized).

(c) The expected volatility over the life to maturity of the option. That is, we consider a dynamic σ_t that is computed as

$$\sigma_t^2 = \frac{1}{T-t} E_t \left[\int_t^T v_u du \right].$$

Such a choice can be justified by observing that, after conditioning on the path of the volatility, in the case of no correlation ($\rho = 0$), the underlying asset S is log-normal, hence we have:

$$\begin{aligned} H_t = E_t[H] = E_t \left[E_t \left[H \,\middle|\, \int_t^T v_u du \right] \right] \\ = E_t \left[C_t^{bs} \left(\frac{1}{T-t} \int_t^T v_u du \right) \right] \approx C_t^{bs}(\sigma_t^2) \end{aligned}$$

where $C_t^{bs}(\sigma^2)$ represents the Black-Scholes price at time t as a function of the variance σ^2. The accuracy of the last approximation depends on the linearity of $C_t^{bs}(\sigma^2)$ and is higher for at-the-money options.

For the Heston model we have

$$\sigma_t^2 = c_0(t,T) + c_1(t,T) v_t,$$

where

$$\begin{aligned} c_0(t,T) &= \theta \left(1 - \frac{1 - e^{-\kappa(T-t)}}{\kappa(T-t)} \right), \\ c_1(t,T) &= \frac{1 - e^{-\kappa(T-t)}}{\kappa(T-t)}. \end{aligned}$$

Then, plugging into (4.30), we get

$$\begin{aligned} a(z,t) &= \ln(z) + \left(-\frac{1}{2} z + \frac{1}{2} z^2 \right) (T-t) c_0(t,T); \\ b(z,t) &= \left(-\frac{1}{2} z + \frac{1}{2} z^2 \right) (T-t) c_1(t,T) 1_v + (z-1) 1_y \end{aligned}$$

where 1_v has the analogous meaning as 1_y.

3. Locally variance-optimal strategy in case trading may be done in continuous time. In this case we have:

$$\begin{aligned}
\theta_t^\star &= \frac{d\langle S, H\rangle_t}{d\langle S, S\rangle_t} \\
&= \Delta_t^H + \mathcal{V}_t^H \frac{d\langle S, v\rangle_t}{d\langle S, S\rangle_t} \\
&= \Delta_t^H + \frac{\rho\sigma}{S_t}\mathcal{V}_t^H
\end{aligned} \tag{4.31}$$

Observe that, when $\rho = 0$, this is equal to the model Delta. In our setting, we have

$$\theta_t^\star = \int_{\mathcal{C}} (z + \rho\sigma\beta(z1_y, t, T) \cdot 1_v) e^{\alpha(z1_y,t,T)+(\beta(z1_y,t,T)-1_y)\cdot X_t} \Pi(dz)$$

that is an affine representation. Since $P = Q$, this would be also globally optimal if trading were allowed in continuous time (for more details on this we refer to Černý and Kallsen (2008)).

Notice that the common practice of employing the implied volatility as σ_t^{bs}, re-computed at each trading date, does not fall in the picture above, because the implied volatility is a function of y. Hence such a strategy is not representable as in (4.30). It would still have an integral representation, but in general not with an exponentially affine integrand, so it does not have an affine representation.

Since we supposed that $P = Q$, the Beta strategy defined in (3.21) is the globally variance-optimal strategy for the discrete time case. We can compute it at each trading date t_k, but, since it does not have an affine representation, we cannot use Theorem 3.1 to compute the expected value and the variance of the related hedging error. As far as we know, there is no result in the literature that can be used to make such a computation. However, we use it as a benchmark for the various strategies at time $t = 0$ (see Figure 3, top panel). Also, it converges to strategy θ_t^* above as the trading interval goes to zero.

Since S is a martingale, in all cases the expected value of the hedging error does not depend on the strategy adopted, but only on the difference between the price H_0 of the option at time 0 and the initial capital c (given also that the risk-free rate is 0). For instance, if H is a liability, the expected final gain would be the extra money $c - H_0$ invested in the strategy. To evaluate the performance of the various strategies in this case, we look at the variance of the related hedging errors. In the general case, when S is not a martingale, one may look at performance indices like the Sharpe ratio.

For the rest of the subsection, we discuss how to check the technical assumptions on the process and on the strategies described above. Assumptions (2.8), (2.9) on the process and that on the function g in (3.15) related to the strategy can be formulated in terms of the set $M(T)$. Because of (4.29), Conditions (2.8), (2.9) simply translate to $(2R, 0)$ and $(2, 0)$ belonging to $M(T)$. As for the strategy, since the second component of X is positive, the assumption on function g is implied by

$$2\mathrm{Re}(b(z,t_k)) + \beta(21_y, t_k, t_j) \in [\omega_1, \omega_2] \times (-\infty, \omega^\star] \subset M(T) \text{ for all } z \in \mathcal{C} \tag{4.32}$$

where ω_1,ω_2 and $\omega^\star$ are real numbers. Indeed, if this is the case,

$$|g(z,t_k,t_j)| \leq E\left[e^{\omega_1 y_{t_k} + \omega^\star v_{t_k}} + e^{\omega_2 y_{t_k} + \omega^\star v_{t_k}}\right]$$

Since the right hand side is finite, we get that $g(z,t_k,t_j)$ is bounded. Of course, this is true for any two-dimensional stochastic volatility diffusion process. We will exploit such formulation in the following

Lemma 4.1 *In the Heston model, if Assumptions (2.8) and (2.9) are satisfied, the functions $g(z,t_k,t_j)$ in (3.15) are bounded on $\mathcal{C}$, for all $k = 0,\ldots,N-1$ and $j = k, k+1$, if the point $C(T) = (2R, \omega^\star(T)) \in M(T)$, where*

1. *for Strategy 1 and 3*

$$\omega^\star(T) = 2\max\{0, \beta_2(R1_y, 0, T)\} + \max\{0, \beta_2(21_y, 0, T)\}$$

2. *for Strategy 2.c*

$$\omega^\star(T) = \frac{1-e^{-\kappa T}}{2\kappa} + \max\{0, \beta_2(21_y, 0, T)\}$$

3. *for Strategies 2.a and 2.b*

$$\omega^\star(T) = \max\{0, \beta_2(21_y, 0, T)\}$$

Proof. For the Heston model and any strategy considered, the function $g(z, t_k, t_j)$ in (3.15), for $z \in \mathcal{C}$, is

$$E\left[e^{(2\mathrm{Re}(b(z,t_k))+\beta(21_y,t_k,t_j))\cdot X_{t_k}}\right] =$$
$$E\left[e^{(2\mathrm{Re}(b_1(z,t_k))+2)y_{t_k}+(2\mathrm{Re}(b_2(z,t_k))+\beta_2(21_y,t_k,t_j))v_{t_k}}\right] =$$
$$E\left[e^{2Ry_{t_k}+(2\mathrm{Re}(b_2(z,t_k))+\beta_2(21_y,t_k,t_j))v_{t_k}}\right]$$

where

$$b_2(z, t_k) = \begin{cases} \beta_2(z1_y, t_k, T) & \text{Strategies 1 and 3} \\ (z^2 - z)\frac{(T-t_k)c_1(t_k,T)}{2} & \text{Strategy 2.c} \\ 0 & \text{Strategies 2.a and 2.b} \end{cases}$$

From Jensen's inequality and the affine representation property ((3.18) or (2.3)), one has

$$Re(\beta_2(z1_y, t, T)) \leq \beta_2(Re(z)1_y, t, T)$$

therefore, for Strategies 1 and 3, $2Re(b_2(z, t_k)) \leq 2\beta_2(R1_y, t_k, T)$. Case 1 and 3 follow from the monotonicity in $T - t$ of function $\beta_2(z1_y, t, T)$ and the initial condition $\beta_2(z1_y, T, T) = 0$. Case 2 follows from a straightforward computation. □

Condition on the function f in (3.14) can be checked by inspection case by case. In our experience, $Re(a(z, t))$ is bounded from above decaying to minus infinity as the absolute value of the imaginary part of z goes to infinity, for all the strategies examined. For the arguments above and Lemma 4.1, the verification of the other conditions boils down to the immediate check that three points belong to $M(T)$. Moreover, this check does not depend on the trading dates. Alternatively, skipping Lemma 4.1, one might check Condition (4.32) for all k and $j = k, k + 1$, which may be easily computed numerically for any set of parameters.

Figure 1 provides an illustration of the results of this subsection. We consider the set of parameters $\theta = 0.05$, $\kappa = 3$, $\sigma = 0.5$ and $\rho = 0.5$ for the top panel and $\rho = -0.9, 0, 0.9$ for the bottom panel. For a call option, we illustrate the case $R = 2$ (any $R > 1$ would be sufficient). Points A and B in Figure 1 are related to Assumptions (2.8) and (2.9), while points C are those of Lemma 4.1 (although it is hard to spot due to the scale of the figure, the points C do not exactly coincide because they depend on T and on ρ). Since all the points belong to $M(T)$, all technical assumptions are satisfied by these parameters and the different maturities T.

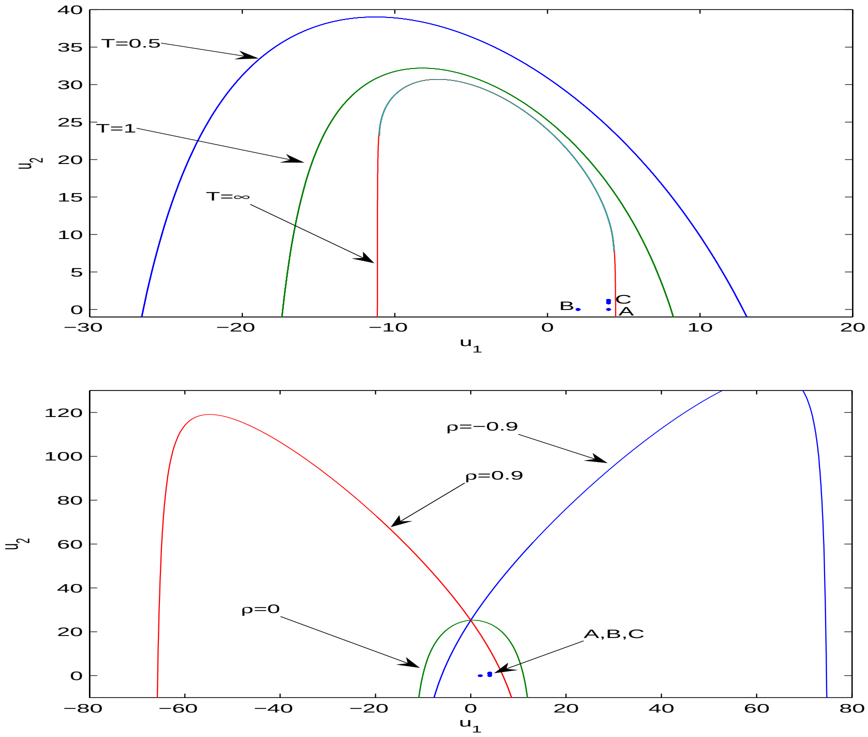

Figure 1 *Domains of existence of the exponential moments $M(T)$ for Heston model with parameters $\mu = 0$, $\theta = 0.05$, $\kappa = 3$, $\sigma = 0.5$. The top panel shows the boundaries of $M(T)$ for $T = 0.5, 1, \infty$ and $\rho = 0.5$. The bottom panel represents the boundaries of $M(1)$ for $\rho = -0.9, 0, -0.9$. Points A, B represent Conditions (2.8) and (2.9), while C is given by Lemma 4.1 related to the function g in (3.14). Since the points belong to the domains, all the conditions are satisfied.*

4.3 Applications

We show several applications of our results: an analysis of the variance as the number of trading dates increases, a study of the influence of the correlation coefficient ρ on the different strategies, and a measure of the impact of moneyness. We conclude by comparing the two-dimensional inversion algorithm to a Monte Carlo approach.

For the first illustration, we take model parameters $v_0 = 0.0102$, $\theta = 0.0190$, $\kappa = 6.21$, $\sigma = 0.61$, $\rho = -0.7$ from Duffie, Pan and Singleton (2000). Notice that this set does not satisfy the non-absorbing condition $2\kappa\theta > \sigma^2$ and this would give serious numerical problems in Monte Carlo simulations. After checking that technical assumptions are satisfied, we consider an at-the-money European call option with maturity $T = 0.25$ years and we suppose that the initial price of the underlying asset is $S_0 = 100$. We take the number of equally spaced rebalancing intervals to be $N = [1, 3, 6, 12, 24, 36, 48, 60, 72]$ and compare the model Delta (Strategy 1), the Black-Scholes Delta with volatility equal to the expected volatility over the life to maturity of the option (Strategy 2.c) and strategy $\theta^\star$ (Strategy 3). The results are shown in Figure 2. Here, as in the illustrations below, it appears that the model Delta gives a higher variance than the Black-Scholes Delta. Strategy $\theta^\star$ outperforms the other two, as it is expected since it would be variance-optimal in the limit as the trading intervals go to zero. To have an idea of the different performances, we report that the differences between the standard deviations, in percentage terms with respect to that of $\theta^\star$, increase with N, going from about 12% to about 20% for the Black-Scholes Delta and from about 30% to about 40% for the model Delta. Hence, the risk of implementing a Black-Scholes hedging strategy for stochastic volatility models is not negligible, but it is still better than using the model Delta. This will be analyzed further in the next experiment.

To analyze the effect of the coefficient ρ on the hedging error we consider an at-the-money call option with maturity $T = 0.5$ years, with an initial price of the underlying asset $S_0 = 100$. From here on, we set $v_0 = 0.05$, $\theta = 0.05$, $\kappa = 3$, $\sigma = 0.5$. For this set of parameters, the ATM implied volatility is between about 21% and 22% for all ρ, while the smile of volatility may range from about 14% to 27% for extreme values of ρ. Strategies compared are: the model Delta (Strategy 1), the Black-Scholes Delta with volatility equal to the expected volatility over the life to maturity of the option (Strategy 2.c) and strategy θ^*_t (Strategy 3). We fix the number of trading dates to be $N = 6$.

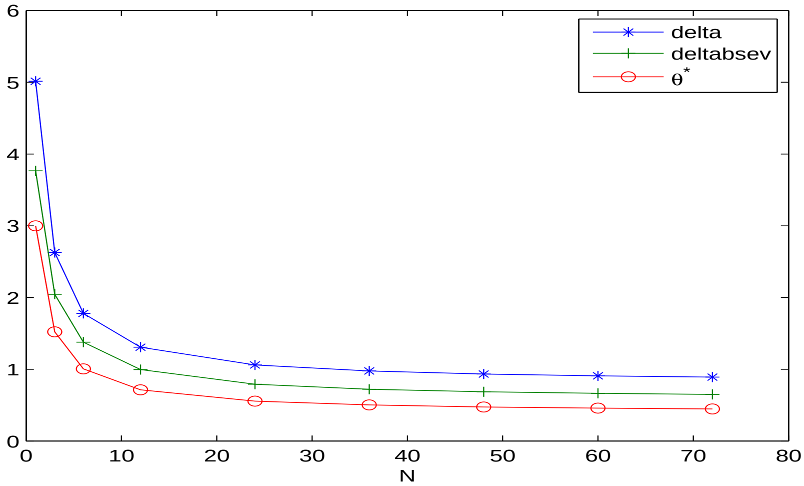

Figure 2 *Variances of hedging strategies for an at-the-money European call option with maturity* $T = 0.25$ *as a function of the number of hedging intervals* N*. The current value of the underlying is* 100*. Hedging ratio: model Delta (delta, Strategy 1), Black-Scholes Delta with expected volatility (deltabsev, Strategy 2.c),* $\theta^\star$ *(Strategy 3).*

The results are represented in Figure 3. The first panel shows the hedge ratios at time $t = 0$ for the three strategies and also for the (variance-optimal in this case) Beta strategy (3.21) as a function of ρ. We notice the difference, due to the discretization, between the optimal Beta strategy and $\theta^{\star}$. We also see that the Black-Scholes Delta is always in between the Beta strategy and the model Delta. In the second panel, showing the variances of the hedging error, we see that the closer the hedge ratio is to the Beta strategy, the smaller is the variance. Hence, we see that the model Delta performs worse than the Black-Scholes delta for all the values of the correlation different from zero. This fact is somehow surprising, but not completely new, in fact, it has been observed by Poulsen et al. (2009) who, in an extensive empirical investigation on the hedging strategies over three different markets reported that "(...) Black-Scholes delta hedges (i.e. the ones where everything is just calculated at implied volatility) perform better than standard delta hedges based on genuine stochastic volatility models". A similar result, in case of a particular Lévy process, was also shown in a numerical experiment by Denkl et al. (2011).

To explain the behaviour of the hedge ratios for at-the-money call options as the correlation factor ρ changes, represented in the top panel of Figure 3, remind that the ρ affects the skewness of the distribution of returns, as an increasing positive (negative) value of ρ makes the right (left) tail of the distribution fatter. Hence the Beta strategy, that is the optimal strategy in this case, increases the exposure to the underlying as ρ increases. The Black-Scholes Delta strategy is independent of ρ, therefore the hedge ratio is constant with respect to ρ. Poulsen et al. (2009) explained the relation between model Delta and Black-Scholes delta. We repeat here their argument for the sake of completeness. Both the Heston call price C^H and the Black-Scholes price C^{bs} satisfy the Euler equation,

$$
\begin{aligned}
C^H &= S\Delta^H + K\frac{\partial C^H}{\partial K} \\
C^{bs} &= S\Delta^{bs} + K\frac{\partial C^{bs}}{\partial K}
\end{aligned}
$$

Moreover, since the option is at-the-money, C^H and C^{bs} are almost equal. The partial derivative of the call price with respect to K is equal to the probability of the option to end up in the money, that is an increasing function of ρ for the Heston case while it is independent of ρ for the Black-Scholes model. Since Δ^H and Δ^{bs} are close to each other when ρ is zero, we conclude that Heston Delta Δ^H is greater than Δ^{bs} for negative ρ and smaller than it when ρ is positive.

To compare the sensitivities of the variance of the error to ρ, we computed the derivative with respect to ρ (third panel). We remark that such a computation is possible thanks to the semi-explicit formulas of Theorem 3.1. In fact, the figure was obtained by taking the derivative with respect to ρ of the integrands in (3.23), (3.24) with a software for symbolic calculus. We observe that all derivatives are almost linear in ρ, that the Black-Scholes Delta is less sensitive, and therefore more robust to mispecification of ρ, and that the model Delta and $\theta^\star$ have an opposite behaviour.

Figure 4 shows the effect of moneyness on the various strategies, also including the two remaining choices for the Black-Scholes Delta, cases 2.a and 2.b, for ρ =(-0.9,0,0.5). We consider various strike prices from 80 to 120. The best performance in all cases is obtained by the strategy $\theta^\star$. For $\rho = 0$, the different strategies have similar performances, even though those of Strategies 2.a and 2.b are slightly worse, as shown in middle panel. The differences are more significant for high values of the correlation coefficient, either positive (bottom panel) or negative (top panel). When ρ is not zero, all the Black-Scholes Deltas (Strategies 2) perform better than the model Delta (except for very far out-of-the-money options where the Black-Scholes Delta 2.c has higher variance than the model Delta). We see that, in terms of comparative performances of the different strategies, we have analogous situations for positive and negative correlation coefficients. The difference between positive and negative ρ is in the general shape of the variance as a function of the strike: while for negative ρ all strategies tend to have a decreasing variance in the out-of-the-money part, for positive ρ this is not the case. This is due to the positively proportional relation in the model between the correlation coefficient and the skewness of the return, showing the impact of the latter on the variance of the hedging error.

Finally, we perform a test of the two-dimensional inversion algorithm adopted to evaluate the formula for the variance in Theorem 3.1. We consider an at-the-money European call with maturity $T = 0.25$ when the present value of the underlying is $S_0 = 100$, hedged with the model Delta strategy. We fix the correlation coefficient to be $\rho = -0.5$ and the number of trading dates to be $N = 1$. Using the approach described at the end of Section 3 and implementing the one-dimensional inversion algorithm, we obtain a standard deviation of the hedging error of 3.5582 (and a price of the option of 4.2959). Then we run a Monte Carlo simulation with different number

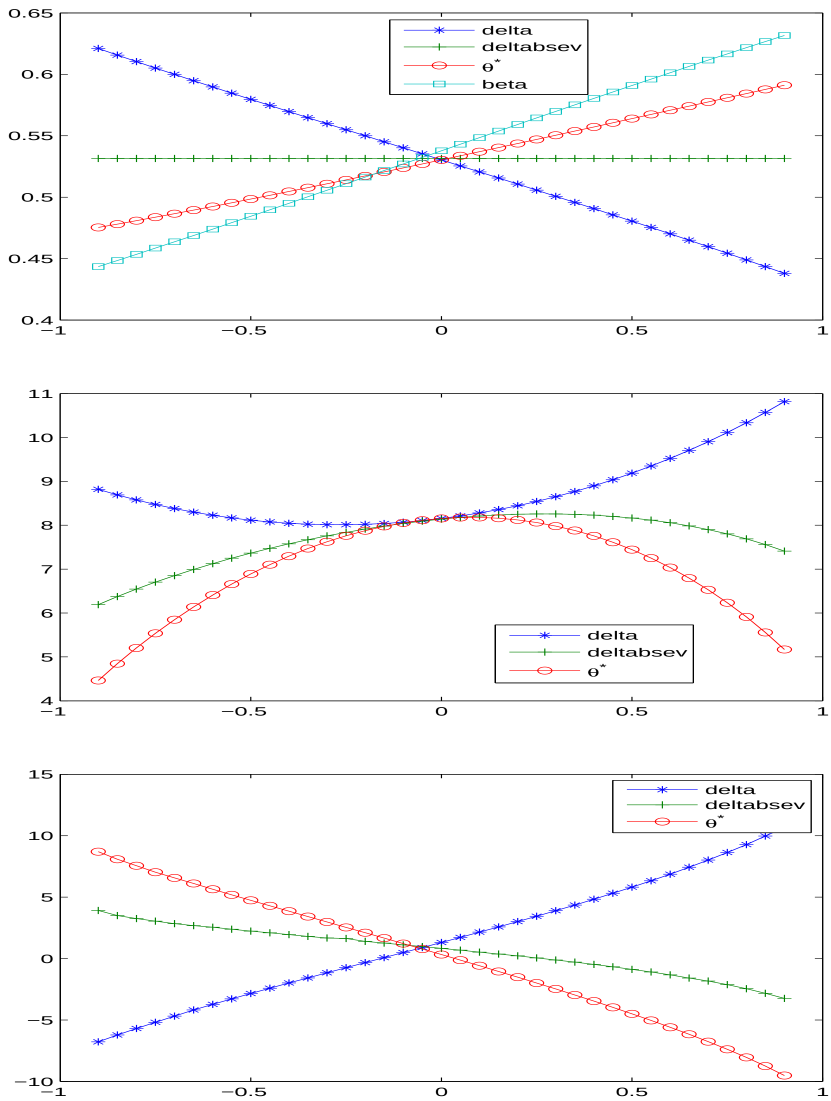

Figure 3 *Hedging strategies for a European call option with maturity* $T = 0.5$ *as a function of the correlation coefficient* ρ*. The current value of the underlying is* 100*. Comparison of three strategies: model Delta (delta, Strategy 1), Black-Scholes Delta with expected volatility (deltabsev, Strategy 2.c),* $\theta^\star$ *(Strategy 3). Top: hedge ratios, which also shows the Beta strategy (3.21). Middle: variances of hedging errors. Bottom: derivatives of the variances with respect to* ρ*.*

Nsim	time (secs)	MC 95% c.i.	2-dim alg	n	time (secs)
1000	2.8	$(3.4220, 3.7358)$	3.5497	40	1.1
10000	26.5	$(3.5193, 3.6183)$	3.5573	100	3.2
100000	312.7	$(3.5441, 3.5753)$	3.5582	200	8.2

Table 1: Standard deviation of the hedging error when the number of trading dates is $N = 1$. A one-dimensional integration scheme gives the value 3.5582. Comparison between the two-dimensional inversion algorithm and Monte Carlo simulation in terms of computational times. The Table reports, from left to right, the number of simulations $Nsim$, the time in seconds for the simulations, the 95% confidence interval, the value returned by the two-dimensional inversion algorithm, the value of the main parameter n of the algorithm and its computational time in seconds. The other parameters of the algorithm are set to $A_1 = A_2 = 30$, $l_1 = l_2 = 1$, $m = 40$ (see Angelini and Herzel (2009)). The strategy is the model Delta for an at-the money call option with maturity $T = 0.25$ and initial value $S_0 = 100$.

of simulations 1000, 10000 and 100000, implementing a simple discretization scheme with time step $\Delta t = 10^{-4}$, and we compute a 95% confidence interval for the standard deviation. The two-dimensional inversion algorithm is implemented with parameters $A_1 = A_2 = 30$, $l_1 = l_2 = 1$, $m = 40$ and for three values of $n = 40, 100, 200$ (see Angelini and Herzel (2009)). Table 1 shows a comparison of the results together with the computational times in seconds. Computations were made on a laptop with dual processor at 2.26GHz. We see that the values returned by the two-dimensional algorithm always fall in the intervals and that the computational times are less. We remark again that we have no definite report in terms of numerical precision from this test and also that in the Monte Carlo simulations we detected some negative values for the variance process.

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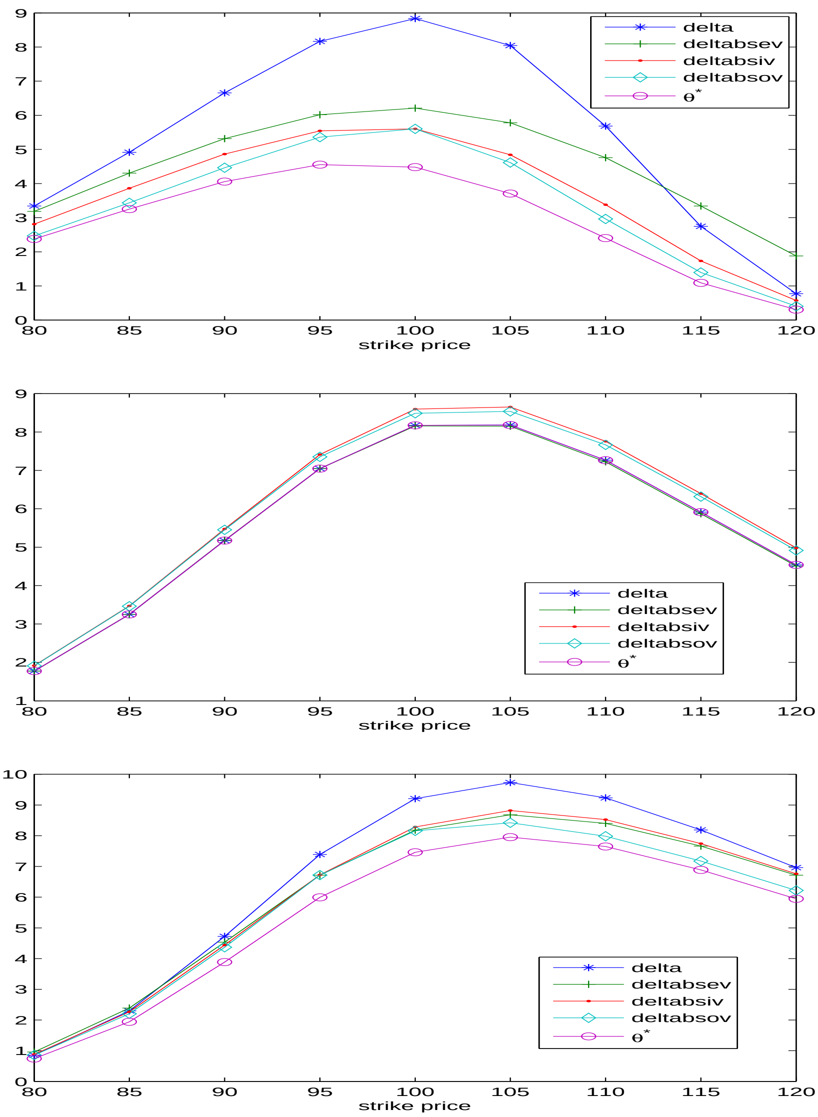

Figure 4 *Variances of hedging strategies for European call options with maturity* $T = 0.5$ *as a function of the strike price. The current value of the underlying is* 100. *Hedging ratio: model Delta (delta, Strategy 1), Black-Scholes Delta with expected volatility (deltabsev, Strategy 2.c), implied volatility (deltabsiv, Strategy 2.a), optimal volatility (deltabsov, Strategy 2.b),* $\theta^\star$ *(Strategy 3). The correlation coefficient are -0.9 (top panel), 0 (middle panel), 0.5 (lower panel).*

6 Appendix

Proof of Theorem 3.1. Given (3.22), applying Fubini's Theorem because of (3.17), (2.10) and the finiteness of measure Π, we have,

$$
\begin{aligned}
& E\left[H - \sum_{k=1}^{N} \vartheta_{t_{k-1}} \Delta S_k\right] = \\
= \quad & \int_{\mathcal{C}} \left\{ E\left[e^{zy_T}\right] - \sum_{k=1}^{N} E\left[e^{a(z,t_{k-1})+b(z,t_{k-1})\cdot X_{t_{k-1}}} \Delta S_k\right] \right\} \Pi(dz) = \\
= \quad & \int_{\mathcal{C}} \left\{ E\left[e^{zy_T}\right] - \sum_{k=1}^{N} e^{a(z,t_{k-1})} \times E\left[e^{b(z,t_{k-1})\cdot X_{t_{k-1}}} \left(e^{y_{t_k}} - e^{y_{t_{k-1}}}\right)\right] \right\} \Pi(dz) = \\
= \quad & \int_{\mathcal{C}} \left\{ E\left[e^{zy_T}\right] - \sum_{k=1}^{N} e^{a(z,t_{k-1})} \times \left(E\left[e^{b(z,t_{k-1})\cdot X_{t_{k-1}} + 1_y \cdot X_{t_k}}\right] \right.\right. \\
& \left.\left. - \; E\left[e^{(b(z,t_{k-1})+1_y)\cdot X_{t_{k-1}}}\right] \right) \right\} \Pi(dz) = \\
= \quad & \int_{\mathcal{C}} \left\{ \phi(z1_y, X_0, 0, T) - \sum_{k=1}^{N} e^{a(z,t_{k-1})} \times \right. \\
& \left. \left(\phi_2(b(z,t_{k-1}), 1_y, X_0, 0, t_{k-1}, t_k) - \phi(b(z,t_{k-1}) + 1_y, X_0, 0, t_{k-1}) \right) \right\} \Pi(dz)
\end{aligned}
$$

which is (3.23). To prove (3.24) we need to compute

$$
\begin{aligned}
& E\left[(H - \sum_{k=1}^{N} \vartheta_{t_k} \Delta S_k)^2\right] = \\
& = E\left[\int_{\mathcal{C}} \left(e^{zy_T} - \sum_{k=1}^{N} e^{a(z,t_{k-1})+b(z,t_{k-1})\cdot X_{t_{k-1}}} \Delta S_k \right) \Pi(dz) \right. \\
& \left. \int_{\mathcal{C}} \left(e^{wy_T} - \sum_{k=1}^{N} e^{a(w,t_{k-1})+b(w,t_{k-1})\cdot X_{t_{k-1}}} \Delta S_k \right) \Pi(dw) \right] = \\
& = E\left[\int_{\mathcal{C}} \int_{\mathcal{C}} \left(e^{zy_T} - \sum_{k=1}^{N} e^{a(z,t_{k-1})+b(z,t_{k-1})\cdot X_{t_{k-1}}} \Delta S_k \right) \times \right. \\
& \left. \left(e^{wy_T} - \sum_{k=1}^{N} e^{a(w,t_{k-1})+b(w,t_{k-1})\cdot X_{t_{k-1}}} \Delta S_k \right) \Pi(dz)\Pi(dw) \right]
\end{aligned}
$$

From (3.17) and (2.10), and using the Cauchy-Schwartz inequality, we can again apply Fubini's Theorem, so we get

$$E\left[(H-\sum_{k=1}^{N}\vartheta_{t_k}\Delta S_k)^2\right]=$$
$$=\int_{\mathcal{C}}\int_{\mathcal{C}}E\left[\left(e^{zy_T}-\sum_{k=1}^{N}e^{a(z,t_{k-1})+b(z,t_{k-1})\cdot X_{t_{k-1}}}\Delta S_k\right)\times\right.$$
$$\left.\left(e^{wy_T}-\sum_{k=1}^{N}e^{a(w,t_{k-1})+b(w,t_{k-1})\cdot X_{t_{k-1}}}\Delta S_k\right)\right]\Pi(dz)\Pi(dw).$$

Let us compute all the expectations needed:

$$E\left[e^{(z+w)y_T}\right]=\phi((z+w)1_y,X_0,0,T).$$

$$E\left[e^{zy_T}\sum_{k=1}^{N}e^{a(w,t_{k-1})+b(w,t_{k-1})\cdot X_{t_{k-1}}}\Delta S_k\right]=$$
$$=\sum_{k=1}^{N}e^{a(w,t_{k-1})}E\left[e^{b(w,t_{k-1})\cdot X_{t_{k-1}}}e^{zy_T}\Delta S_k\right]=$$
$$=\sum_{k=1}^{N}e^{a(w,t_{k-1})}\left(E\left[e^{b(w,t_{k-1})\cdot X_{t_{k-1}}+y_{t_k}+zy_T}\right]-E\left[e^{b(w,t_{k-1})\cdot X_{t_{k-1}}+y_{t_{k-1}}+zy_T}\right]\right)=$$
$$=\sum_{k=1}^{N}e^{a(w,t_{k-1})}\left(E\left[e^{b(w,t_{k-1})\cdot X_{t_{k-1}}+1_y\cdot X_{t_k}+z1_y\cdot X_T}\right]-E\left[e^{(b(w,t_{k-1})+1_y)\cdot X_{t_{k-1}}+z1_y\cdot X_T}\right]\right)=$$
$$=\sum_{k=1}^{N}e^{a(w,t_{k-1})}\times$$
$$(\phi_3(b(w,t_{k-1}),1_y,z1_y,X_0,0,t_{k-1},t_k,T)-\phi_2(b(w,t_{k-1})+1_y,z1_y,X_0,0,t_{k-1},T))=$$
$$=v_2(w,z).$$

The expectation

$$E[e^{wy_T}\sum_{k=1}^{N}e^{a(z,t_{k-1})+b(z,t_{k-1})\cdot X_{t_{k-1}}}\Delta S_k]$$

is obtained as above after interchanging w with z.

The last term is

$$E\left[\sum_{j=1}^{N}\sum_{k=1}^{N} e^{a(w,t_{j-1})+b(w,t_{j-1})\cdot X_{t_{j-1}}}\Delta S_j e^{a(z,t_{k-1})+b(z,t_{k-1})\cdot X_{t_{k-1}}}\Delta S_k\right]$$
$$=\sum_{j=1}^{N}\sum_{k=1}^{N} e^{a(w,t_{j-1})}e^{a(z,t_{k-1})}E\left[e^{b(w,t_{j-1})\cdot X_{t_{j-1}}+b(z,t_{k-1})\cdot X_{t_{k-1}}}\Delta S_j\Delta S_k\right].$$

Expanding the products

$$\Delta S_j\Delta S_k$$

one gets $v_4(w,z)$. □

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