

Ample divisors on the blow up of \mathbb{P}^3 at points

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We prove a theorem that gives a necessary and sufficient condition for certain divisors on the blow up of \mathbb{P}^3 at points in general position to be ample, which extends a theorem of G. Xu [Xu] on the blow up of \mathbb{P}^2 .

1 Introduction

In this note we will prove a theorem on divisors on the blow up of \mathbb{P}^3 at points which extends a theorem of G. Xu [Xu] on the blow up of \mathbb{P}^2 . The central idea of the proof works for any dimension and therefore opens the doors to a generalization of the theorem to higher dimension, once one overcomes certain technical difficulties that arise. Basically we will give a new proof of Xu's theorem that works also for \mathbb{P}^3 and so in most of this note n will be either 2 or 3.

Theorem. *Let $n = 2$ or 3 . Let \mathbb{P}^n be the projective space over the field of complex numbers. Let p_1, \dots, p_k be k points in \mathbb{P}^n in general position and let $\pi : X \rightarrow \mathbb{P}^n$ be the blow up of \mathbb{P}^n at p_1, \dots, p_k with exceptional divisors E_1, \dots, E_k . Let $H = \pi^* \mathcal{O}_{\mathbb{P}^n}(1)$. Then, if $d \geq d_0(n)$, the divisor $L = dH - \sum_{i=1}^k E_i$ is ample if and only if $L^n > 0$, i.e. $d^n > k$, where $d_0(2) = 3$, $d_0(3) = 5$.*

Remark 1.1. In the case of \mathbb{P}^2 we obtain the same bound as in Xu's, which is sharp. For \mathbb{P}^3 we believe, following a conjecture, that the theorem holds

for $d \geq 3$.

Xu's proof is based on an estimate for the self-intersection of moving singular curves in \mathbb{P}^2 . The same estimate was obtained also by Ein and Lazarsfeld in the context of Seshadri constants on smooth surfaces [EL] and used by Küchle to prove the above theorem in the case of a smooth surface [Ku].

The basic idea of the proof comes from an example of R. Miranda regarding ample divisors on a smooth surface with arbitrarily small Seshadri constant [L, §5]. The strategy is as follows: we will use the fact that ampleness is an open condition to reduce to the case when the points are part of the base locus of a $(n-1)$ -dimensional general linear system of hypersurfaces of degree d . The corresponding line bundle L on the blow up at the entire base locus is nef and this will imply that the divisor obtained by blowing down some of the exceptional loci is ample. For this last step we will use in a determinant way the fact that the fibres of the morphism to \mathbb{P}^{n-1} determined by L are irreducible, under the stated conditions on d . We will prove this fact in Proposition 2.1 which concerns curves in \mathbb{P}^3 which are complete intersection. The problem is to give a lower bound for the codimension, in the space of such curves, of the space of reducible ones. We emphasize that it is this technical proposition that gives, apart from the lower bound on d , that may be not sharp, but reasonable, the restriction for the dimension of the projective space for which the theorem holds. The rest of the proof works for any dimension and therefore we will present it as much as possible in its generality in section 3. We would like to mention that it is actually possible to prove Proposition 2.1, and therefore the theorem, also in the cases $n = 4$ and 5 , in a way very similar to the case $n = 3$ [A]. However we will not show it in this note.

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2 Preliminary material and lemmas

Let \mathbb{P}^N be the projective space parametrizing hypersurfaces in \mathbb{P}^n of degree d . So $N = N(d, n) = \binom{n+d}{n} - 1$. If F is such a hypersurface, we will denote by $[F]$ the corresponding point in \mathbb{P}^N . Recall that there is an action of $PGL(N)$ on \mathbb{P}^N . If σ is an element of $PGL(N)$ and Λ is a linear subspace of \mathbb{P}^N , which corresponds to an element of a Grassmannian $Gr(\mathbb{P}^k, \mathbb{P}^N)$, we will denote by Λ^σ the linear subspace obtained by letting σ act on Λ . We will say that a property holds for a general linear subspace Λ of \mathbb{P}^N if it holds outside a union (possibly countable) of proper subvarieties of $Gr(\mathbb{P}^k, \mathbb{P}^N)$. We will denote by Σ_d the locus of singular hypersurfaces which is an irreducible subvariety of \mathbb{P}^N of codimension one.

We will also be dealing with curves which are complete intersection of elements of \mathbb{P}^N . These are parametrized by an open subset of the Grassmannian $Gr(\mathbb{P}^{n-2}, \mathbb{P}^N)$ (of course for $n = 2$ this is just $\mathbb{P}^{N(d,2)}$). We will denote by $[l_C] \in Gr(\mathbb{P}^{n-2}, \mathbb{P}^N)$ the point corresponding to a curve C . Also we have $\dim Gr(\mathbb{P}^k, \mathbb{P}^r) = (k+1)(r-k)$.

Let now

$$NL_{d,3} = \left\{ \begin{array}{l} \text{smooth surfaces } S \subset \mathbb{P}^3 \text{ of degree } d \\ : Pic(S) \text{ is not generated by the hyperplane class} \end{array} \right\}.$$

$NL_{d,3}$ is called the *Noether-Lefschetz locus* and may be viewed as a subset of $\mathbb{P}^{N(d,3)}$. This locus is pretty well understood, at least as far as we are concerned here (see for example [GH2]): $NL_{d,3}$ is a countable union of quasi-projective algebraic varieties. The Noether-Lefschetz theorem asserts that, for $d \geq 4$, the general surface of degree d in \mathbb{P}^3 has Picard group generated by the hyperplane section, i.e. is not contained in $NL_{d,3}$. In other words the theorem says that the codimension of all the irreducible components N_i of $NL_{d,3}$ is at least one. What we will need is an explicit Noether-Lefschetz theorem (see [G] for a nice proof of it) giving a precise bound for the codimension of any of the N_i . I was informed by the referee that the theorem was also obtained independently by C. Voisin.

Theorem. *For $d \geq 4$, the codimension of any irreducible component of $NL_{d,3}$ in \mathbb{P}^N is at least $(d-3)$.*

The main technical result we need is:

Proposition 2.1. *Let $n = 2, 3$. Assume $d \geq d_0(n)$, with $d_0(2) = 3$ and $d_0(3) = 5$. Let Λ be a general linear subspace of \mathbb{P}^N of dimension $(n - 1)$ whose base locus consists of d^n points. Then, for every $(n - 1)$ linearly independent elements $[F_1], \dots, [F_{n-1}]$ of Λ , the intersection $F_1 \cap \dots \cap F_{n-1}$ is a curve and is irreducible.*

In the case $n = 2$ this is just saying that every element of a general pencil of curves of degree d is irreducible for $d \geq 3$.

Remark 2.1. Proposition 2.1, and therefore the theorem, would be proven for any dimension if one found a good bound for the codimension, in the space of curves in \mathbb{P}^n which are a complete intersection of hypersurfaces of degree d , of the space of reducible ones. It is easy to conjecture that this codimension should be at least $(n - 1)(d - 1)$, being this the codimension of such curves which have a line as a component. This conjecture would imply Proposition 2.1 with $d_0(n) = 3$ and for any n .

Proof. The proof in the case $n = 2$ is rather easy. The set Σ_d of singular curves of degree d in \mathbb{P}^2 is an irreducible subvariety of codimension one. The reducible curves are union of closed subvarieties all lying in Σ . It is then enough to notice that, for $d \geq 3$, there are irreducible singular curves to establish that the codimension of the reducible ones is at least two and therefore conclude that a general pencil does not contain reducible curves.

For $n = 3$ we take $d \geq 5$ and we choose a general plane $\Lambda \subset \mathbb{P}^N$, with $N = \binom{d+3}{3} - 1$, whose base locus consists of d^3 points and so that no member of it is reducible. A curve C which is a complete intersection of two elements of Λ corresponds to a line $l_C \subset \Lambda$, i.e. to an element $[l_C] \in Gr(\mathbb{P}^1, \Lambda) \subset Gr(\mathbb{P}^1, \mathbb{P}^N)$. First we have the following:

Claim 2.1. For any l_C in Λ , we can find a smooth surface S with $[S] \in l_C$ such that $[S]$ is not in $NL_{d,3}$.

For this we need the following lemmas:

Lemma 2.1. *The intersection of Λ with Σ_d does not contain any line $l_C \subset \Lambda$. In particular, for any curve C which is a complete intersection of elements of*

Λ there exist at most finitely many surfaces S with $[S] \in l_C$ that are singular.

Proof. Σ_d is an irreducible subvariety of codimension one and high degree of \mathbb{P}^N , and hence, for a general Λ , $\Sigma_d \cap \Lambda$ is an irreducible curve of degree strictly greater than one. \square

Lemma 2.2. *For $d \geq 5$, the intersection of $\Lambda \subset \mathbb{P}^N$ with the Noether-Lefschetz locus consists of at most countably many points. This is to say that there are at most countably many smooth surfaces S with $[S] \in \Lambda$ and with $Pic(S)$ not generated by the hyperplane section.*

Proof. We apply Kleiman's Transversality Theorem [H, Thm. III.10.8] for the action of $PGL(N)$ on \mathbb{P}^N to a plane $\Lambda \subset \mathbb{P}^N$ and the closure $\overline{N_i}$ of each irreducible component N_i of $NL_{d,3}$. \square

Proof of Claim 2.1. For any l_C in Λ , by Lemma 2.2 there are at most countably many surfaces S with $[S] \in l_C$ and $Pic(S)$ not generated by the hyperplane section and by Lemma 2.1 there are at most finitely many singular ones. It is then possible to find a smooth S with $Pic(S)$ generated by the hyperplane section. \square

Now we can prove Proposition 2.1 for $n = 3$.

Proof of Proposition 2.1. Let C be any curve which is complete intersection of elements of Λ . We need to prove that C is irreducible. Choose a surface S as in Claim 2.1. Pick another surface T with $[T] \in l_C$ so that $C = S \cap T$. If C were reducible, say $C = \cup C_i$, then any $C_i \subset S$ would be a complete intersection $S \cap T_i$, with T_i a hypersurface in \mathbb{P}^3 of degree less than d , since $Pic(S)$ is generated by the hyperplane section. But then

$$C = S \cap T = \cup(S \cap T_i) = S \cap (\cup T_i)$$

This means that $[\cup T_i] \in l_C \subset \Lambda$. But Λ misses reducible surfaces, so C has to be irreducible. \square

As mentioned in the introduction, it is possible to prove Proposition 2.1 in the case $n = 4$ and 5 along the same line as for $n = 3$, using a generalized explicit Noether-Lefschetz theorem, due to S. Kim [Ki], regarding smooth surfaces which are complete intersection of hypersurfaces [A]. However we decided not to include it here, but we just would like to say that this type

of proof does not work in dimension greater than 6 because the Noether-Lefschetz theorem does not give any information about singular surfaces and it is not possible anymore to ensure the existence of a smooth surface containing the curve like in Lemma 2.1.

Another tool for the proof of the theorem is the fact that ampleness is an open condition. We will state this very well known fact in the form of:

Proposition 2.2. *Let $\mathcal{L} \rightarrow T$ be a flat family of line bundles over a flat family $\mathcal{X} \rightarrow T$ of projective varieties of dimension n . Then the set*

$$\{t \in T : \mathcal{L}_t = \mathcal{L}|_{X_t} \text{ is ample on } X_t\}$$

is open in T .

(See [A] for a proof of it).

3 Proof of Theorem

The central idea of the proof works for any dimension and we will present the argument as much as possible in its generality. The restriction on the dimension comes uniquely from Proposition 2.1, to ensure the irreducibility of the fibers of the morphism μ below.

We are given k points in general position in \mathbb{P}^n and we need to prove that the divisor $dH - \sum_{i=1}^k E_i$ is ample when $d^n > k$ for $d \geq d_0(n)$. Clearly this condition is necessary. We consider a general linear system Λ of hypersurfaces of degree d and dimension $(n-1)$ and we let p'_1, \dots, p'_{d^n} be the base locus of Λ . Let $\pi' : X' \rightarrow \mathbb{P}^n$ be the blow up at k of these points p'_1, \dots, p'_k , $H' = \pi'^* \mathcal{O}_{\mathbb{P}^n}(1)$ and E'_1, \dots, E'_k the exceptional divisors. We will prove that $dH' - \sum_{i=1}^k E'_i$ is ample and the theorem will follow from Proposition 2.2.

To do so we consider the blow up $\nu : Z \rightarrow \mathbb{P}^n$ at the whole base locus p'_1, \dots, p'_{d^n} and set $H = \nu^* \mathcal{O}_{\mathbb{P}^n}(1)$, E_i for $i = 1, \dots, k$, F_i for $i = k+1, \dots, d^n$

the exceptional divisors of ν and $L = dH - \sum_{i=1}^k E_i - \sum_{i=k+1}^{d^n} F_i$. We have the following diagram:

$$\begin{array}{ccc} Z & \xrightarrow{\delta} & X' \\ \nu \swarrow & & \searrow \pi' \\ & \mathbb{P}^n & \end{array}$$

where δ is the blow down of F_{k+1}, \dots, F_{d^n} . Now L is globally generated and therefore is nef. We will show that, for $d \geq d_0(n)$, this implies that $dH' - \sum_{i=1}^k E'_i$ is ample. Now we have a morphism $\mu : Z \rightarrow \mathbb{P}^{n-1}$ whose fibers are curves which are complete intersection of elements of Λ . Moreover, if C is a curve,

$$L \cdot C = 0 \iff \mu(C) \text{ is a set of points.}$$

By Proposition 2.1, for $n = 2, 3$ and $d \geq d_0(n)$, by taking Λ sufficiently general, we may arrange for these fibers to be all irreducible. What we need is the following:

$$(L + \sum_{i=k+1}^{d^n} F_i)^m \cdot Y_m \geq 0 \tag{3.1}$$

$$(L + \sum_{i=k+1}^{d^n} F_i)^m \cdot Y_m = 0 \iff Y_m \subset F_i \text{ for one } i = k+1, \dots, d^n, \tag{3.2}$$

where Y_m is any irreducible subvariety of dimension m of Z , for any $0 < m < n$.

Notice that:

$$L + \sum_{i=k+1}^{d^n} F_i = dH - \sum_{i=1}^k E_i = \delta^*(dH' - \sum_{i=1}^k E'_i).$$

(3.1) and (3.2) prove, by Nakai's criterion and the projection formula, that $dH' - \sum_{i=1}^k E'_i$ is ample on X' .

To simplify notation let $L' = L + \sum_{i=k+1}^{d^n} F_i$. From here on we restrict to $n = 2$ and 3 because we will use Proposition 2.1. If Proposition 2.1 were proven for any n we would proceed by induction on m , using the same arguments. Here we just have to check (3.1) and (3.2) for $m = 1$ in the case $n = 2$ and for $m = 1$ and 2 in the case $n = 3$.

Let $m = 1$ so that Y_1 is a curve. If $Y_1 \subseteq F_i$ for one i , then, being L' trivial on F_i , $L' \cdot Y_1 = 0$ and, if $Y_1 \not\subseteq F_i$ for all i , then, being $L \cdot Y_1 \geq 0$ (L is nef) and $\sum_{i=k+1}^{d^n} F_i \cdot Y_1 \geq 0$, $L' \cdot Y_1 \geq 0$. This proves (3.1) and the easy direction of (3.2). For the other direction suppose $L' \cdot Y_1 = 0$ but $Y_1 \not\subseteq F_i$ for all i . But then, being $\sum_{i=k+1}^{d^n} F_i \cdot Y_1 \geq 0$, $L \cdot Y_1$ has to be zero and therefore Y_1 is a component of a fiber of μ . Since every fiber is irreducible, Y_1 is exactly a fiber and so it meets all the exceptional divisors of ν . Therefore $\sum_{i=k+1}^{d^n} F_i \cdot Y_1$ is strictly positive which is a contradiction.

In the case $n = 3$ we need also to consider any irreducible subvariety Y_m of Z of dimension $m = 2$. As before, if $Y_m \subseteq F_i$ for some i , then $L' \cdot Y_m = 0$. If $Y_m \not\subseteq F_i$ for all i , then we write:

$$L'^m \cdot Y_m = L \cdot L'^{m-1} \cdot Y_m + \sum_{i=k+1}^{d^n} F_i \cdot L'^{m-1} \cdot Y_m.$$

Now $\sum_{i=k+1}^{d^n} F_i \cdot Y_m$ has dimension strictly less than m (or is empty) and $L \cdot Y_m$ can be represented by a cycle of dimension strictly less than m (or empty) because L is globally generated. So both $L \cdot L'^{m-1} \cdot Y_m$ and $\sum_{i=k+1}^{d^n} F_i \cdot L'^{m-1} \cdot Y_m$ are greater or equal than zero by the previous step and so is $L'^m \cdot Y_m$. For the remaining direction of (3.2) suppose $L'^m \cdot Y_m = 0$ while $Y_m \not\subseteq F_i$ all i . Then, since as before $\sum_{i=k+1}^{d^n} F_i \cdot L'^{m-1} \cdot Y_m \geq 0$, we have $L \cdot L'^{m-1} \cdot Y_m = 0$. By the previous step, this implies that $L \cdot Y_m \subseteq F_i$ for some i . We claim that this

is a contradiction, i.e. we claim that, if $Y_m \not\subseteq F_i$ for any i , then $L \cdot Y_m \not\subseteq F_i$ for any i . Indeed, if $Y_m \not\subseteq F_i$ for any i , then $\nu_*(Y_m) \subseteq \mathbb{P}^3$ has dimension 2. Consider a divisor $D \in |\Lambda|$. By Bezout's Theorem $\dim(\nu_*(Y_m) \cdot D) \geq 1$. Also

$$\dim(\nu_*(Y_m) \cdot D) = \dim(\nu_*(Y_m \cdot \nu^* D)) = \dim(\nu_*(Y_m \cdot L)).$$

So $\dim(\nu_*(Y_m \cdot L)) \geq 1$ and therefore $Y_m \cdot L \not\subseteq F_i$ for any i . \square

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