

On the Effect of Skewness and Kurtosis Misspecification on the Hedging Error

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Abstract

Using a result in Angelini and Herzel (2009a), we measure, in terms of variance, the cost of hedging a contingent claim when the hedging portfolio is re-balanced at a discrete set of dates. We analyze the dependence of the variance of the hedging error on the skewness and kurtosis as modeled by a Normal Inverse Gaussian model. We consider two types of strategies, the standard Black-Scholes Delta strategy and the locally variance-optimal strategy, and we perform some robustness tests. In particular, we investigate the effect of different types of model misspecification on the performance of the hedging, like that of hedging without taking skewness into account. Computations are performed using a Fast Fourier Transform approach.

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1 Introduction

One of the main issues that quantitative methods and models for financial markets have to deal with is incompleteness, namely that it is not possible to perfectly replicate a contingent claim or a future liability with a portfolio of traded assets. Models for financial markets englobe incompleteness by introducing more sources of risk than traded assets. For instance, in the class of exponential Lévy models, which have recently been extensively studied for pricing and hedging derivatives, this is achieved by adding jumps to the dynamics of the driving process (see for instance Cont and Tankov 2004a and Cont et al. 2005). While most of these models assume that trading is possible in continuous time, in practice only trading in discrete time is possible, and this introduces another font of incompleteness that may be called "discretization error", even in models that would otherwise be complete as the Black-Scholes one. In the framework of exponential Lévy models, Tankov and Voltchkova (2009) show an asymptotic analysis of the discretization error and Angelini and Herzel (2009a) compute the variance of the hedging error of discrete time strategies. In that class of models, log-returns are assumed to be independent and identically distributed and the properties of the distribution, like skewness and fatness of the tails, obviously affect the risk related to the position to be hedged. In his book on incomplete markets, Černý (2009) proposes an approximating formula for the variance of the hedging error of Black-Scholes Delta hedging in an exponential Lévy process in terms of the first four moments of the distribution of the log-returns. In Černý's Formula, the hedging portfolio is re-adjusted at a discrete set of dates and the standard deviation of the log-returns is used as volatility parameter for the Black-Scholes Delta. Such a formula therefore provides a quick way to measure the discretization error. Moreover, it deals with another type of error because, while the driving distribution is supposed to have non-zero skewness and excess of kurtosis, the strategy under exam is conceived for a normal distribution of the log-returns. This may be called hedging under model misspecification. Another instance of such is when a strategy, in principle coherent with the model, is implemented with incorrect estimates of the parameters of the model. For instance, Angelini and Herzel (2009a) study the effect of a misspecified volatility parameter on the hedging error in the Black-Scholes framework. We will extend such analysis under different hypothesis on the driving distribution, in particular in presence of skewness and kurtosis.

The problem of hedging derivatives in incomplete markets has been studied by many authors (see for instance Schweizer 1995, Hubalek et al. 2006, Cont et al. 2005, Bertsimas et al. 2001), following the seminal paper of Föllmer and Sondermann (1986). The main approach to the problem is that of determining a strategy which minimizes the variance of the hedging error, which we will refer to as the optimal strategy. Generally more feasible to compute, is the strategy that minimizes the variance of the cost of local portfolio adjustments, which we will call the locally optimal strategy.

However, the Black-Scholes Delta strategy is still very popular among practitioners. In a different model setting than the Black-Scholes one, this choice is not theoretically coherent, but it can nevertheless be followed, with two possibilities: using the volatility of the process of the log-returns of the underlying, like in Černý's Formula, or the Black-Scholes implied volatility of the option. A feasible alternative strategy is the model Delta, namely the derivative of the model price of the derivative with respect to the underlying. As pointed out by Tankov (2007), in a model with jumps this choice is not optimal, since it does not take into account the risk coming from the jumps, but only that coming from infinitesimal movements. Moreover, in some models, it may even not exist. A sensible solution is to compute the optimal strategy.

The aim of the paper is to study the influence of the realized distribution of the driving process on the hedging error, comparing the Black-Scholes Delta and the optimal strategy, under different properties of the distribution and under model misspecification. More precisely, we compute the hedging strategy under a given hypothesis for the first four moments, which we will think of as estimated. Then we let the underlying asset evolve driven by a distribution of the same type, but with different moments, which we think of as the realized distribution. In particular, we wish to study the case of incorrect estimates of skewness and kurtosis, for instance in case that the hedge is constructed supposing zero skewness while the realized distribution presents some. For the analysis we consider the well known Normal Inverse Gaussian model because of the close relation between its parameters and such moment indices. We mention that another way of looking at the problem is to think of the hedging strategy constructed using risk neutral parameter estimates, namely coming from calibration to option prices, like in the case of delta type hedging ratio. The related probability distribution is quite different from that inferred from historical data, in particular has very different skewness and kurtosis, as shown in Carr et al. (2002). Nevertheless, both

the pricing measure and the historical measure may be used to forecast the distribution of the underlying process (for a discussion see Aït-Sahalia 2000, Cont et al. 2005, Cont and Tankov 2004b and 2006).

The computational method we adopt is based on the representation of the payoff of the claim as an inverse Laplace transform. This approach was proposed by Hubalek et al. (2006) in order to efficiently compute the optimal strategy and its variance. It was used by Angelini and Herzel (2009a) to evaluate hedging strategies satisfying a compatibility condition which is met by various important strategies, like delta-type ones. With such a tool we will also assess the validity of Černý's Formula.

The Laplace approach provides with a framework suitable for managing derivatives, namely for pricing, computing hedging ratios and measuring the cost of hedging. For pricing and calibration purposes, as well as for computation of Greeks, a popular numerical method adopted by financial institutions is the Fast Fourier Transform (FFT) algorithm, proposed by Carr and Madan (1999). Another contribution of the paper is to implement the FFT algorithm to compute the expectation and the variance of the hedging error and compare it with the two-dimensional inversion algorithm adopted in Angelini and Herzel (2009a).

2 Theoretical and Computational Framework

2.1 Model for the underlying

Let $t_n = n\Delta t$, $n = 0, 1, \dots, N$, be a set of trading dates, with $\Delta t = T/N$ and T a time horizon. Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \in (0,1,\dots,N)}, P)$ denote a filtered probability space and let $X = (X_{t_n})_{n=0,1,\dots,N}$ be a real-valued process with independent and stationary increments. More precisely:

1. X is adapted to the filtration $(\mathcal{F}_n)_{n \in (0,1,\dots,N)}$,
2. $X_0 = 0$,
3. $\Delta X_n = X_{t_n} - X_{t_{n-1}}$ has the same distribution for $n = 1, \dots, N$,
4. ΔX_n is independent from \mathcal{F}_{n-1} for $n = 1, \dots, N$.

We model the price process $S = (S_n)_{n=0,1,\dots,N}$ of a non dividend paying stock at time t_n as follows

$$S_n = S_0 e^{X_{t_n}} . \tag{2.1}$$

We assume that $E[S_1^2] < \infty$ so that the moment generating function

$$m(z) = E[e^{zX_{\Delta t}}] \quad (2.2)$$

is defined at least for complex z with $0 \leq \text{Re}(z) \leq 2$. Moreover, we exclude the case when S is a deterministic process. We also suppose that there exists a deterministic risk-free asset.

We are interested in studying the effect of the distribution of the log-returns, in particular that due to its skewness and kurtosis, on the performance of the hedging. In principle, the methodology developed in Angelini and Herzel (2009a) can be applied for any Lévy model, and in Angelini and Herzel (2009b) the authors extend their approach also to the case of Affine models. As in Hubalek et al. (2006) and in Denkl et al. (2009), in all the applications we develop in Section 3, we consider the Normal Inverse Gaussian (NIG) model, proposed for financial applications in Barndorff-Nielsen (1998). The NIG model was extensively studied in literature both for its quite realistic path properties and because, in spite of its few parameters, it is able to adequately fit the market implied volatility surfaces. See for example Jensen et al. (2001) and Carr et al. (2003). In the NIG model, the Lévy process X is an infinite activity and infinite variation pure jump process. It can be obtained subordinating a Brownian motion of volatility σ and drift θ with an independent Inverse Gaussian process I_t^ν of variance rate ν . Moreover a constant deterministic drift μ can be considered: $X_t = \mu t + \theta I_t^\nu + \sigma W_{I_t^\nu}$. Its moment generating function is:

$$m(z) = e^{\left(z\mu + \frac{1}{\nu} - \frac{1}{\nu}\sqrt{1 - z^2\sigma^2\nu - 2z\theta\nu}\right)\Delta t}. \quad (2.3)$$

A very attractive feature of it is that, apart from the drift parameter, it depends only on three parameters (σ, θ, ν) . These parameters are related in a simple way to the variance $\text{Var}(X_t)$, the skewness $\text{Skew}(X_t)$ and the excess of kurtosis $\text{ExKurt}(X_t)$ of the distribution of X_t , which makes the NIG model particular suitable to the purposes of this work.

The cumulants of the process at time t are (Cont and Tankov 2004a)

$$\begin{aligned} k_1 &= (\mu + \theta)t \\ k_2 &= (\sigma^2 + \theta^2\nu)t \\ k_3 &= (3\sigma^2\theta\nu + 3\theta^3\nu^2)t \\ k_4 &= (3\sigma^4\nu + 15\theta^4\nu^3 + 18\sigma^2\theta^2\nu^2)t \end{aligned} \quad (2.4)$$

from which we get:

$$\begin{aligned}
Var(X_t) &= k_2 = (\sigma^2 + \theta^2\nu)t \\
Skew(X_t) &= \frac{k_3}{k_2^{3/2}} = \frac{3\theta\nu}{\sqrt{Var(X_1)}} \frac{1}{\sqrt{t}} \\
ExKurt(X_t) &= \frac{k_4}{k_2^2} = \left(3\nu + 12 \frac{\theta^2\nu^2}{Var(X_1)} \right) \frac{1}{t}
\end{aligned}
\tag{2.5}$$

The variance is given by a linear combination of σ^2 and $\theta^2\nu$, while the sign of skewness is given by the sign of θ . Parameter ν mostly affects the excess of kurtosis. So for example, if $\theta = 0$, the distribution is symmetric, the variance of X_1 is simply σ^2 and the excess of kurtosis is just 3ν . If θ is different from zero σ, θ and ν combine in a non linear way to give the variance, skewness and excess of kurtosis of the distribution, and thus they can no more be thought as three independent parameters for the determination of the moments. Nevertheless, it is possible to invert the relations in (2.5) to find how the model parameters depend on the moments:

$$\begin{aligned}
\sigma^2 &= Var(X_1) \frac{3ExKurt(X_1) - 5Skew(X_1)^2}{3ExKurt(X_1) - 4Skew(X_1)^2} \\
\theta &= \frac{3Skew(X_1)\sqrt{Var(X_1)}}{3ExKurt(X_1) - 4Skew(X_1)^2} \\
\nu &= \frac{1}{3}ExKurt(X_1) - \frac{4}{9}Skew(X_1)^2
\end{aligned}
\tag{2.6}$$

In order to have $\nu > 0$ and $\sigma^2 > 0$, the skewness and excess of kurtosis have to satisfy the following relation

$$ExKurt(X_1) \geq \frac{5}{3}Skew(X_1)^2, \tag{2.7}$$

so that, at least with our parametrization of the model, it is not possible to achieve distributions with arbitrary skewness and excess of kurtosis.

2.2 Integral representation of payoffs

In their works Hubalek et al. (2006) and Angelini and Herzel (2009a) consider European derivative securities on the stock with price S , with maturity T

and payoff $H = f(S_T)$, where the function $f : (0, \infty) \rightarrow \mathbb{R}$ is of the form

$$f(S) = \int S^z \Pi(dz). \quad (2.8)$$

Π is a complex measure on a strip in the complex plane $\{z \in \mathbb{C} : R' \leq \operatorname{Re}(z) \leq R\}$, where R' and R are real numbers. For this, we need to make some further integrability assumptions on the price process, namely $E[S_1^{2R'}] < \infty$ and $E[S_1^{2R}] < \infty$.

Note that, when $R = R'$, Equation (2.1) simply means that the payoff function can be written as an inverse Laplace transform. For instance, the payoff of an European call option with strike price $K > 0$ and maturity T , can be written as

$$(S - K)^+ = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} S^z \frac{K^{1-z}}{z(z-1)} dz, \quad (2.9)$$

for an arbitrary $R > 1$ and for each $S > 0$. For other examples of integral representation of payoffs, like that of the put, the power call and the digital option, see Hubalek et al. (2006).

2.3 Hedging strategies

Let $\vartheta = (\vartheta_n)$, for $n = 1, \dots, N$, be a hedging strategy. The random variable ϑ_n is interpreted as the number of shares of the underlying asset held from time $(n-1)\Delta t$ up to time $n\Delta t$. Suppose moreover that the strategy is admissible, that is a predictable process such that the cumulative gains are square-integrable (Hubalek et al. 2006, Schweizer 1995).

Following Angelini and Herzel (2009a), we consider strategies of the form

$$\vartheta_n = e^{-r(N-n+1)\Delta t} \int f_n^\vartheta(z) S_{n-1}^{z-1} \Pi(dz), \quad (2.10)$$

for all $n = 1, \dots, N$, where $f_n^\vartheta(z)$ is a deterministic function of the complex variable z . We will briefly show some examples of such strategies.

Starting from the integral representation of payoff (2.8), one can compute the price of the claim at time t_n just performing the expectation in a risk neutral world conditional to \mathcal{F}_n and discounting at the risk-free rate r

$$C_n = e^{-r(N-n)\Delta t} E_n \left[\int S_N^z \Pi(dz) \right].$$

Exchanging the expected value integral with the Laplace integral by Fubini's Theorem and using the fact that the log-returns are i.i.d. random variables, one gets the integral representation of the price value:

$$C_n = e^{-r(N-n)\Delta t} \int S_n^z m(z)^{N-n} \Pi(dz).$$

Previous price formula holds for any model satisfying the conditions of Section 2.1 and 2.2 and whose moment generating function $m(z)$ is known.

From the above pricing formula, it comes natural to take the derivative with respect to S_n

$$\Delta_n = \frac{\partial C_{n-1}}{\partial S_{n-1}} = e^{-r(N-n+1)\Delta t} \int z m(z)^{N-n+1} S_{n-1}^{z-1} \Pi(dz). \quad (2.11)$$

This is the so called Delta strategy and it depends on the model chosen to construct it. Let us note that in principle such a strategy can be constructed in a model which is different from that of the underlying. As an example of that, consider the Black-Scholes Delta, one of the most common hedging strategy among practitioners. This is just an instance of Equation (2.11) with

$$m(z) = e^{\left(\left(r - \frac{\sigma^2}{2}\right)z + \frac{\sigma^2}{2}z^2\right)\Delta t}$$

the moment generating function of the normal distribution. Even though the Black-Scholes Delta strategy is conceived for a log-normal process, it can be applied in our setting where the underlying moves following the NIG dynamics. It is common market practice to implement such a strategy fixing the volatility parameter $\tilde{\sigma}$ to the implied volatility of the claim to be hedged and varying this parameter according to the implied volatility observed at each time of re-balancing. Note that, since the implied volatility depends on the price of the underlying, such a Black-Scholes Delta strategy does not satisfy Equation (2.10). In our model setting, the log-returns of the underlying are i.i.d., hence we choose to set the volatility $\tilde{\sigma}$ to the standard deviation of their distribution (as for instance in Černý 2009 and Angelini and Herzel 2009a).

Another instance of strategy is the Delta within the model. In our setting, it means that the moment generating function in (2.11) is that of the NIG distribution. However, as pointed out by Tankov (2007), in a model with jumps this choice is not optimal, since it does not take into account the risk coming from the jumps, but only that coming from infinitesimal movements.

This is confirmed by our numerical experiences and those in Denkl et al. (2009).

An appropriate choice in this setting is the locally optimal hedging ratio, that is the strategy which minimizes the variance of the next period costs and whose formal definition can be found in Schweizer (1995). Such a strategy is given in Theorem 2.1 in Hubalek et al. (2006) for the case a zero risk-free rate, that is equivalent to consider the process of discounted prices rather than prices. Writing the explicit dependence on r , this is

$$\xi_n = e^{-r(N-n+1)\Delta t} \int f_n^\xi(z) S_{n-1}^{z-1} \Pi(dz), \quad (2.12)$$

where $f_n^\xi(z) = g(z)h(z)^{N-n}e^{r\Delta t}$, with

$$g(z) = \frac{m(z+1) - m(1)m(z)}{m(2) - m(1)^2},$$

$$h(z) = m(z) - (m(1) - e^{r\Delta t})g(z).$$

Most important in this context is the strategy that minimizes the expected squared value of the total hedging error, called the optimal strategy. It is well known that such a strategy exists (Schweizer 1995) and it is computed together with its variance in Hubalek et al. (2006) using the Laplace transform approach. Such an optimal strategy is another important example of a strategy that, in general, does not satisfy Equation (2.10). We recall that, if the discounted price process is a martingale, then this coincides with the locally optimal strategy and it therefore satisfies Equation (2.10).

2.4 Measurement of hedging error

Let $\vartheta = (\vartheta_n)$, for $n = 1, \dots, N$, be a hedging strategy. The cumulative gains in the presence of a money market account can be simply obtained capitalizing (or discounting) up to the same date at the risk-free interest rate all the cash-flows coming from portfolio re-adjustments. We choose to capitalize the cash-flows up to the date of maturity of the claim to hedge. Thus, the cumulative gains from the strategy at time $T = N\Delta t$ are:

$$G_N(\vartheta) = \sum_{k=1}^N \vartheta_k e^{r(N-k+1)\Delta t} (S_k e^{-r\Delta t} - S_{k-1}),$$

and the resulting final hedging error is:

$$\varepsilon(\vartheta, c) = H - e^{rT}c - G_N(\vartheta),$$

where c is the initial capital invested in the strategy.

The strategy ϑ determines a unique self-financing portfolio. The hedging error $\varepsilon(\vartheta, c)$ is nothing but the net loss-gain one can have at maturity if one starts with the initial capital c and follows the strategy.

Consider a European contingent claim whose payoff H may be written as in (2.8) and a given strategy $\vartheta = (\vartheta_n)$ satisfying Equation (2.10), with initial capital c . Angelini and Herzel (2009a), in Theorem (3.1), computed the expected values of the hedging error and of the squared hedging error as

$$E[\varepsilon(\vartheta, c)] = \int S_0^z e(z) \Pi(dz) - e^{rT}c \quad (2.13)$$

with

$$e(z) = \left[m(z)^N - (e^{-r\Delta t}m(1) - 1) \sum_{k=1}^N f^\vartheta(z)_k m(z)^{k-1} \right] \quad (2.14)$$

and

$$E[\varepsilon(\vartheta, 0)^2] = \int \int S_0^{y+z} V(y, z) \Pi(dz) \Pi(dy), \quad (2.15)$$

with

$$V(y, z) = (v_1(y, z) - v_2(y, z) - v_3(y, z) + v_4(y, z)) \quad (2.16)$$

where

$$\begin{aligned} v_1(y, z) &= m(y+z)^N, \\ v_2(y, z) &= \sum_{n=1}^N f_n^\vartheta(y) m(y+z)^{n-1} m(z)^{N-n} (e^{-r\Delta t}m(z+1) - m(z)), \\ v_3(y, z) &= \sum_{n=1}^N f_n^\vartheta(z) m(y+z)^{n-1} m(y)^{N-n} (e^{-r\Delta t}m(y+1) - m(y)), \end{aligned}$$

$$\begin{aligned}
v_4(y, z) &= (e^{-2r\Delta t}m(2) - 2e^{-r\Delta t}m(1) + 1) \sum_{n=1}^N f_n^\vartheta(y) f_n^\vartheta(z) m(y, z)^{n-1} + \\
&+ (e^{-r\Delta t}m(1) - 1) \sum_{j < n} \sum_{n=2}^N f_j^\vartheta(y) f_n^\vartheta(z) m(y)^{n-1-j} m(y+z)^{j-1} \times \\
&\quad \times (e^{-r\Delta t}m(y+1) - m(y)) + \\
&+ (e^{-r\Delta t}m(1) - 1) \sum_{j < n} \sum_{n=2}^N f_n^\vartheta(y) f_j^\vartheta(z) m(z)^{n-1-j} m(y+z)^{j-1} \times \\
&\quad \times (e^{-r\Delta t}m(z+1) - m(z)).
\end{aligned}$$

The variance of the hedging error is

$$var(\varepsilon(\vartheta, c)) = var(\varepsilon(\vartheta, 0)) = E[\varepsilon(\vartheta, 0)^2] - E[\varepsilon(\vartheta, 0)]^2.$$

We reported the result here for convenience of the reader. We also made explicit the dependence on the risk-free rate r .

We shall apply the results in the case of the NIG model, performing the computation of the expectations of the hedging error and of the squared hedging error given in (2.13) and (2.15) with a one-dimensional and a two-dimensional Fast Fourier Transform (FFT) machinery, an algorithm proposed for financial applications in Carr and Madan (1999) to determine call and put option prices.

In Appendix A we show some details on the implementation of the FFT approach to our cases. Moreover we show the comparison with the algorithm used in Angelini and Herzel (2009a).

3 Applications

The main focus of this work is to analyze the effect of the skewness and excess of kurtosis of the log-returns of the process on the performances of hedging for different hedging strategies. As an application of the Laplace/FFT methodology, we will first assess the validity of the Černý's approximation Formula (Černý 2009) for the variance of the Black-Scholes Delta hedging error with discrete portfolio re-balancing. Then we will compare the variance of the hedging error for the locally optimal strategy and for the Black-Scholes Delta

strategy, for different couples of skewness and excess of kurtosis. Finally we will make some misspecification test, allowing the underlying to evolve according to a distribution that has different moments from those used as input for the strategy. We will not show results about the model Delta (2.11) for the NIG model for the sake of a clear exposition. We report, as already mentioned, that its performance is generally even poorer than that of the Black-Scholes Delta.

We will indicate with \mathcal{R} the collection of 16×9 sets of model parameters, selected to give a wide range of models with different distributional properties. We think of \mathcal{R} as possible realizations of the dynamics of the driving process X . As we are mainly interested to skewness and kurtosis effects, to each set in \mathcal{R} we make correspond the same standard deviation of the log-returns, which, expressed on a yearly basis (i.e. the standard deviation of X_t , for $t = 1$ year), is about 20%. The set of parameters are chosen in such a way that the excess of kurtosis for the underlying distribution varies in the set $\{3.3555, 6.3, 12.6, 18.9, 25.2, 63, 126, 189, 252\}$. At any level of excess of kurtosis, we take 16 levels of skewness going from negative to positive values, paying attention to the bound in (2.7). Hence, for each fixed value of the excess of kurtosis, the values of the skewness considered are different, because the range of possible values of skewness compatible with the bound (2.7) depends on the level of excess of kurtosis. The model parameters corresponding to the different couples of skewness and excess of kurtosis, while keeping the variance fixed, are obtained by relations (2.6). The drift parameter μ is set in such a way that the discounted price of the underlying is a martingale. Thus, differently from the variance, the expected value of the yearly log-returns is not constant, depending on the parameters of the model. However, it remains very close to the value -0.02 , over all the sets of parameters in \mathcal{R} . Values for the skewness and excess of kurtosis are expressed on a daily basis (i.e. as the skewness and the excess of kurtosis of X_t , for $t = 1$ day). For the time conversion we considered a year composed by 252 trading days. Let us remark that the values we used are conceivable in the sample distribution of daily log-returns, as for instance shown in Duffie and Pan (1997).

The choice of a martingale measure provide us with a little simplification because in that case the expected value of the hedging error does not depend on the strategy at all, being just the difference between the expectation of the derivative's payoff and the initial capital invested. For such reason we will consider only the variance of the hedging error without considering at

all its expected value. Moreover, as already mentioned, the locally optimal strategy is also globally optimal and we will therefore call it optimal from now on.

In the following applications, we will consider call options with maturity $T = 0.25$ years, for an initial value of the underlying equal to $S_0 = 100$ and a risk-free rate $r = 0$.

3.1 Dependence on skewness and excess of kurtosis

In his book on incomplete markets (Černý 2009) the author proposes an approximated formula to compute the variance of the hedging error when hedging a call of maturity T , at a discrete set of dates $\{t_n, n = 0, \dots, N\}$, using a standard Black-Scholes Delta strategy implemented at the standard deviation of the log-returns. The formula takes into account the higher moments of the objective distribution, correcting Toft's Formula, Toft (1996), with an additive term that is linear in the skewness and in the excess of kurtosis of the underlying log-returns. We report Černý's Formula for the convenience of the reader, making explicit the dependence on the temporal interval $\Delta t = T/N$, in such a way that the expected value $\tilde{\mu}$, the variance $\tilde{\sigma}^2$, the skewness $Skew$ and the excess of kurtosis $ExKurt$ of the realized log-returns are expressed on a time-unit basis:

$$var(\varepsilon) \approx \frac{1}{4} \tilde{\sigma}^4 \left(\frac{T}{N}\right)^2 S_0^4 \Gamma_0^2 \sum_{n=0}^{N-1} g(t_n) \times \left(2 + \left(\frac{N}{T}\right) ExKurt + 4 \frac{\tilde{\mu}}{\tilde{\sigma}} Skew + 4 \left(\frac{\tilde{\mu}}{\tilde{\sigma}}\right)^2 \left(\frac{T}{N}\right)\right) \quad (3.17)$$

Here Γ_0 is the Gamma of the call computed at time zero at the volatility $\tilde{\sigma}$, while function $g(t)$ is the same as in Angelini and Herzel (2009a). If the underlying is driven by a geometric brownian motion, and neglecting the higher order term $(\tilde{\mu}/\tilde{\sigma})^2$, one recovers Toft's Formula from (3.17). Note that, differently from Toft's Formula, the expression in (3.17) does not go to zero with $1/N$ because of the $ExKurt$ term.

Figure 1 shows the variance of the hedging error, when hedging an ATM call, as the number N of trading dates increases. The underlying is supposed to be driven by the NIG model specified by three sets of parameters in \mathcal{R} , corresponding to the three panels of the figure. In each panel, we compare the

result of the FFT/Laplace computation, when the strategy is the standard Black-Scholes Delta, with the approximating Černý's Formula. The variance of the optimal strategy is also shown.

In the first panel of Figure 1 the model parameters are fixed as in Hubalek et al. (2006) except for the drift. The corresponding (daily) skewness and excess of kurtosis are $[Skew, ExKurt] = [-0.17086, 3.3555]$. In this case, corresponding to small values of the skewness and excess of kurtosis, the difference between the optimal strategy and the Black-Scholes Delta can be considered negligible. Moreover, Černý's Formula gives a good approximation of the variance of the hedging error, at least for N not too small. For example, in the case of $N = 12$, the Laplace/FFT computation gives a result for the variance of hedging error of 1.1862 and Černý's Formula returns 1.1539. The accuracy of Černý's Formula increases for higher values of N . For higher value of the excess of kurtosis as in the second and the third specifications of the model, Černý's approximation does not work as well as in the first one, although for intermediate value of the excess of kurtosis the approximation is much better than the Toft's one. For example, in the second panel the excess of kurtosis is $ExKurt = 25.2$, with the same skewness as above. In that case, for $N = 12$, we have a variance of 2.5560, while Černý's Formula returns 2.9697, that is much better than Toft's approximation that is 0.8728. Also in this case, the optimal strategy has a quite comparable behaviour to that of the Black-Scholes Delta strategy. In the last case, corresponding to $[Skew, ExKurt] = [-1.2421, 126]$, Černý's Formula does not work at all.

Next, we will present a more comprehensive analysis by computing the variance of the hedging error for the optimal strategy and the Black-Scholes Delta, as a function of all the sets of parameters in \mathcal{R} . We fix the number of trading dates to be $N = 12$, corresponding to about a weekly re-balancing. For each set in \mathcal{R} , we implement the corresponding optimal strategy, while the Black-Scholes Delta is implemented at the fixed standard deviation.

The results of this computation, in the case of the optimal strategy, are plotted in Figure 2 for a 10% OTM strike, a 10% ITM strike and in the ATM case. The figure shows the variance of the hedging error as a function of the daily skewness and excess of kurtosis, corresponding to each set of parameters in \mathcal{R} . Note that the standard deviation of the hedging error is of the order of some units. These values of course have to be compared with the price of the issue to be hedged. The relative weight of the error with respect to the price thus depends on the moneyness. The variance of the hedging

error grows like a square root with the excess of kurtosis. In the ATM case, the surface is symmetric with respect to the zero skewness level, while in the ITM (OTM) case the slope is negative (positive) with the skewness as expected. Indeed, given an ITM (OTM) call, the probability that it ends OTM (ITM) is greater, and so it is the risk to hedge it, if the skewness of the log-returns distribution is negative (positive).

As for the Black-Scholes Delta strategy, we report that the surface of the variance of the hedging error is flatter than that in Figure 2 of the optimal one. Another difference is that it is increasing in the skewness direction in the ATM case. The plane defined by Černý's Formula looks like tangent to the surface at the point $[0, 0]$, so that we can say that the formula works for small values of the skewness and of the excess of kurtosis similarly to a first order Taylor expansion.

The relative difference between the standard deviations of the hedging error computed with the two strategies, $(\sigma(\varepsilon)^{LO} - \sigma(\varepsilon)^{BS})/\sigma(\varepsilon)^{LO}$, grows with the level of excess of kurtosis and with the absolute value of the skewness. In most of the cases, it is no more than a few percent points. If we look at the values of the excess of kurtosis up to 126, and at levels of skewness between -4 and 4, this difference is always less than 10%, at least for the ATM and ITM case. In the OTM case, for example a 10% OTM call, it can arrive up to 15%. The main message we get from this analysis is that the hedging performance of the Black-Scholes Delta strategy is quite comparable to that of the optimal strategy, unless the driving distribution has very fat tails and high skewness.

3.2 Model misspecification

This subsection is devoted to the analysis of model misspecification in the following sense. We start with a set of model parameters and compute hedging strategies. Such a set may be thought of as obtained through a calibration procedure, for example to the market implied volatility surface of plain derivatives. Otherwise, it can be estimated from historical log-returns, after a moment matching procedure, or can be built in order to incorporate some personal views about the future distributions. Hence, such parameters and the consequent moments of the distribution can be thought of as estimated. Then we let the underlying evolve under the same model, but with a different set of parameters. The related dynamics of the log-returns may be thought of as the realized ones. For example, in the case of the Black-Scholes model,

this means that one implements the Delta hedging at the implied volatility, while the underlying evolves with a different realized volatility (see for example Angelini and Herzel (2009a) for such an experiment). Here, we let the underlying evolve following the NIG dynamics corresponding to each set of parameters belonging to \mathcal{R} .

Among all the sets in \mathcal{R} , we picked out fifteen as input for the optimal strategy. In particular, we chose three values (low, intermediate and high) for the excess of kurtosis, 3.3555, 18.9 and 126, and then, for each of these levels, we chose five different levels of skewness from negative to positive including zero. Let us indicate with \mathcal{E} such a sub-collection of \mathcal{R} . For each set in \mathcal{E} , we compute the strategy and then the variance of the hedging error of call options when the underlying evolves according to all possible dynamics defined by \mathcal{R} . As in the previous section, we fix the number of trading dates to be $N = 12$.

As already remarked, hedging with a Black-Scholes Delta strategy, even if computed at the realized standard deviation, while the underlying evolves under the NIG model, is an instance of model misspecification as the skewness and the excess of kurtosis are neglected at all. In Section 3.1 we have seen that, for a wide range of values of skewness and excess of kurtosis, the relative difference between the standard deviations of the hedging error of the Black-Scholes Delta strategy and the optimal one is, for most of the cases, below 10%. Now we compare the hedging errors when also the optimal strategy is computed under misspecification of skewness and excess of kurtosis. Such a strategy will be called misspecified optimal strategy. For each set in \mathcal{R} , we take as a term of comparison the minimal standard deviation of the hedging error $\sigma(\varepsilon)^{LO}$ within the model, namely that obtained by using the optimal strategy implemented with the correct parameters. The results are given in terms of the relative increase $\sigma(\varepsilon)^{mis}/\sigma(\varepsilon)^{LO} - 1$, expressed as a percentage, where $\sigma(\varepsilon)^{mis}$ is the standard deviation of the hedging error of the misspecified strategy, which could be the Black-Scholes Delta or the optimal. There are of course some cases where there is not increase, that is when the estimated parameters input for the optimal strategy coincide with the realized ones.

We represent the main results of our analysis in Figures 3 and 4, which correspond to two sets in \mathcal{E} . We show the results for an ATM call as a function of (a subset of) the sets in \mathcal{R} . On the x-axis we plot the corresponding skewness, while the three panels in each figure correspond to different levels of the realized excess of kurtosis, 3.3555, 18.9, 126.

Figure 3 shows the case $[Skew, ExKurt] = [0, 18.9]$ as input for the optimal strategy. Differences here are of the order of few percentage points. For other cases in \mathcal{E} with skewness zero, the pattern is the same as in Figure 3 but the differences may increase. The performance of the misspecified optimal strategy is worse for over-estimated kurtosis.

Figure 4 represents the case where both the estimated skewness and the excess of kurtosis as input of the optimal strategy assume quite high values, namely $[Skew, ExKurt] = [-5.5895, 126]$. It can be the case if one calibrates the model over the market option prices. In fact, as it is pointed out in Carr et al. (2002), at least for equities, the skewness and kurtosis in a risk-neutral world are significantly greater than if they were estimated from time series. In particular, the authors find that the risk-neutral skewness is definitively negative, while the skewness as estimated from historical series can be negative, positive, or very close to zero. In the highest two panels, the kurtosis is highly over-estimated and the Black-Scholes Delta performs much better. As the realized kurtosis gets closer to that estimated, as for instance in the bottom panel, the performance of the misspecified optimal strategy improves. This improvement also depends on whether the skewness, in particular its sign, is correctly estimated. This pattern is confirmed also when the estimated skewness is positive, which we do not show.

As for the differences between the standard deviations of the two strategies with respect to the optimal one, $(\sigma(\varepsilon)^{LOmis} - \sigma(\varepsilon)^{BS})/\sigma(\varepsilon)^{LO}$, we see from Figure 4 that the Black-Scholes Delta may lower the risk of the hedging up to about 40% with respect to the misspecified optimal strategy. We report that, in some extreme cases, this may reach about 60%.

Out of the ATM it is more difficult to recognize a precise pattern. In general, the effect of the misspecification is amplified with respect to the ATM with relative differences of the standard deviation of hedging error that can range from -20% to 80% in the 10% ITM case. The latter value is obtained when one seriously over-estimates the excess of kurtosis and computes the strategy with a highly negative value of the skewness while that realized is positive. In the 10% OTM case, the range is from -25% to 115% , the latter case occurring when one over-estimates the kurtosis and computes the strategy with a highly positive value of the skewness while that realized is negative.

The analysis shows that the performances of the optimal strategy strongly depend on whether one correctly estimates the realized dynamics of the model. Overall, the Black-Scholes Delta strategy is less sensitive to the real-

ized dynamics than the optimal strategy.

Table 1 summarizes the results of the analysis, showing the average and the standard deviation over all sets in \mathcal{R} of the relative increase, expressed as a percentage, in standard deviation of the hedging error $\sigma(\varepsilon)^{mis}/\sigma(\varepsilon)^{LO} - 1$, for each of the sixteen misspecified strategies, the Black-Scholes Delta and the fifteen misspecified optimal strategies corresponding to each set in \mathcal{E} . We show such performance indicators separately for the ATM case (second two columns) and over all the strikes we considered (third two columns). In the first columns we indicate the estimated excess of kurtosis and skewness adopted as input of the strategy. The performances of the Black-Scholes Delta and of the optimal strategy implemented with low values of skewness and kurtosis are very much alike. Instead, adopting a strategy that tries to take into account some extreme tail property of the process may lead to serious errors. Overall, our results indicate that the Black-Scholes Delta is more robust than the optimal strategy.

4 Conclusions

We studied the problem of hedging a contingent claim in incomplete models, when the hedging portfolio is re-balanced at a set of discrete dates. In particular, we measured the final hedging error of a given strategy. We started with contingent claims with payoffs having an integral representation to compute, in semi-explicit way, expectation and variance of the hedging error using results from Angelini and Herzel (2009a). The method, which involves inverting Laplace/Fourier transforms, is implemented using the FFT technique, which is a popular algorithm adopted by financial institutions for pricing and calibration purposes. We analyzed the performance of the hedging error of the Black-Scholes Delta strategy and the locally optimal strategy. We considered the NIG model as it has a great relevance both in academic and in practitioner literature and because, in this model, the relation between its parameters and the moments of the driving distribution is very simple. We first studied the validity of the approximated formula proposed in Černý's book (Černý 2009) for different couples of skewness and excess of kurtosis as the number of trading dates increases. Then we analyzed the robustness of the locally optimal strategy and the Black-Scholes Delta, under skewness and excess of kurtosis misspecification. The analysis shows that, when implementing the optimal strategy under the right realized distribu-

tion, the performances are similar. When the skewness and the excess of kurtosis are incorrectly estimated, overall the Black-Scholes Delta strategy performs better than the misspecified locally optimal. In general, it is less sensitive to the realized distribution than the locally optimal strategy, as in the latter case one has to adequately fit the realized distribution. The effect of misspecification on the performance of the hedging is even stronger in the not ATM cases.

A Appendix: Fast Fourier Transform

We show how to perform the computation of the expectations of the hedging error and of the squared hedging error given in (2.13) and (2.15) using a Fast Fourier Transform algorithm.

We consider contingent claims H as in (2.8) with the measure Π concentrated on the vertical line $R + i\mathbb{R}$ and expressed as

$$\Pi(dz) = \frac{1}{2\pi i} K^{\beta-z} p(z) dz,$$

where β is a real number. This is the case for all payoffs proposed in Hubalek et al. (2006). For example, for a call option $p(z) = 1/z(z-1)$ and $\beta = 1$. Let $s_0 = \log(S_0)$ and $\tilde{k} = \log(K/S_0)$.

The value of R depends on the claim to be hedged and on the model considered. In the case of a call and for a set of different models including NIG, it is known that $R - 1 = 1.5$ is a good choice for pricing purpose (see Carr and Madan 1999) and it turned out to be suitable also for the computation of expected value and variance of the hedging error.

First of all we make a change of variables $z = iu + R$ to express the integrals in terms of Fourier transforms. Then the expected value of the hedging error at the maturity of the option can be written as

$$E[\varepsilon(\vartheta, 0)] = \frac{e^{-\tilde{k}(R-\beta)}}{2\pi} \int_{-\infty}^{\infty} e^{-iu\tilde{k}} \chi(iu + R) du, \quad (\text{A.18})$$

where the integrand is

$$\chi(z) = e^{s_0\beta} e(z) p(z), \quad (\text{A.19})$$

and function $e(z)$ is given in (2.14). The above expectation is a Fourier transform in the normalized log-strike variable \tilde{k} of the kernel of integration χ .

The integral can be taken only on the positive real axis because expectation is a real number, which imposes to the real part of the function χ to be even and the imaginary part to be odd.

The expectation of the squared hedging error is given in terms of a two-dimensional Fourier transform

$$E[\varepsilon(\vartheta, 0)^2] = \frac{e^{-(\tilde{k}_1 + \tilde{k}_2)(R - \beta)}}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-iu_1 \tilde{k}_1 - iu_2 \tilde{k}_2} \mathcal{K}(iu_1 + R, iu_2 + R) du_1 du_2, \quad (\text{A.20})$$

for $\tilde{k}_1 = \tilde{k}_2$, and the integration kernel is

$$\mathcal{K}(z_1, z_2) = e^{2s_0\beta} V(z_1, z_2) p(z_1) p(z_2) \quad (\text{A.21})$$

with function $V(z_1, z_2)$ given in (2.16).

The one-dimensional FFT is an algorithm to efficiently compute sums of the form:

$$\Gamma(l) = \sum_{j=1}^M e^{-i\frac{2\pi}{M}(j-1)(l-1)} X(j) \quad (\text{A.22})$$

for $j, l = 1, \dots, M$. The parameter M has to be a power of 2 in order to make the FFT algorithm work efficiently.

Invoking the trapezoid rule, we can approximate the integrals in (A.18) and (A.20) by sums. Let us start with the expected value. The integral reduces to the following sum

$$E[\varepsilon(\vartheta, 0)] = \frac{e^{-\tilde{k}_l(R - \beta)}}{\pi} \sum_{j=1}^M e^{-i\frac{2\pi}{M}(j-1)(l-1)} \chi(j) \eta \quad (\text{A.23})$$

for $j, l = 1, \dots, M$. The vector $\chi(j)$ is the integrand (A.19) computed at the discrete set of points $z_j = iu_j + R$, with $u_j = \eta(j - 1)$. The parameter η is the lattice spacing of the integration variable and $M\eta$ becomes the effective upper limit of integration. The summation returns a vector of the same dimension of a vector of log-strikes equally spaced with lattice spacing λ , centered around the at the money, namely $\tilde{k}_l = (l - 1 - M/2)\lambda$. In order to get a sum of the form (A.22), the lattice spacings have to be related according to the following relation: $\eta\lambda = 2\pi/M$.

The algorithm depends on two free parameters, M and η , and one has to find the right trade-off between a fine grid for integration and a good grid

around the at-the-money for the strike dimension. Introducing some Simpson's weights into the summation, one can obtain an accurate integration even with a larger value of η , thus leaving the possibility of a finer grid in the strike dimension. The vector to be transformed with FFT algorithm thus becomes

$$X(j) = \chi(iu_j + R) \frac{\eta}{3} [3 + (-1)^j - \delta_{j-1}], \quad (\text{A.24})$$

where δ_n is the Kronecker delta function that is one for $n = 0$ and zero otherwise.

A two-dimensional FFT computes for any two-dimensional complex input array $X(j_1, j_2)$, with $j_{1,2} = 1, \dots, M$, the output array

$$\Gamma(l_1, l_2) = \sum_{j_1=1}^M \sum_{j_2=1}^M e^{-i \frac{2\pi}{M} ((l_1-1)(j_1-1) + (l_2-1)(j_2-1))} X(j_1, j_2), \quad (\text{A.25})$$

for $l_{1,2} = 1, \dots, M$. The trapezoid rule in this case needs to define the following $M \times M$ grid $\{(u_{1,j_1}, u_{2,j_2}) : j_1, j_2 = 1 : \dots, M\}$, with

$$u_{1,j_1} = (j_1 - 1 - M/2)\eta \quad u_{2,j_2} = (j_2 - 1 - M/2)\eta,$$

while the strike grid is $\{(\tilde{k}_{1,l_1}, \tilde{k}_{2,l_2}) : l_1, l_2 = 1 : \dots, M\}$, with

$$\tilde{k}_{1,l_1} = (l_1 - 1 - M/2)\lambda \quad \tilde{k}_{2,l_2} = (l_2 - 1 - M/2)\lambda.$$

As usual, the strike grid and the integration grid are related by relation $\eta\lambda = 2\pi/M$.

In this case, the trapezoid rule applied to the integral in (A.20) gives the following:

$$E[\varepsilon(\vartheta, 0)^2] \approx \frac{e^{-2\tilde{k}_l(R-\beta)}}{(2\pi)^2} \Gamma(l, l), \quad (\text{A.26})$$

where $\Gamma(l, l)$ is just the diagonal part of the two-dimensional array $\Gamma(l_1, l_2)$ that is the output of the two-dimensional FFT algorithm like in (A.25), where the input is the array $X(j_1, j_2)$ given by

$$X(j_1, j_2) = (-1)^{(j_1-1)+(j_2-1)} \mathcal{K}(iu_{1,j_1} + R, iu_{2,j_2} + R)\eta^2,$$

and the kernel \mathcal{K} is defined in (A.21).

We implemented the code in MATLAB, using the MATLAB functions "fft.m" and "fft2.m" to compute respectively the one-dimensional and two-dimensional Fast Fourier Transform. We fixed the number of lattice points and the lattice spacing parameter according to $M = 1024$ and $\eta = 0.25$. This choice was tested to be a good trade-off between accuracy and computational time. The output is for a vector of strikes. The corresponding log-strike spacing is around 2.4544%.

The advantage of using an FFT algorithm instead of an approach based on the inversion of a Laplace transform is that FFT gives results for a whole vector of strikes. Let us concentrate for example on the computation of the expectation of squared hedging error and let us compare the computational time needed to compute such a quantity with the two-dimensional Laplace inversion algorithm, proposed in Choudhury et al. (1994) and used in Angelini and Herzel (2009a), and an FFT algorithm. Consider a call option with maturity $T = 0.25$ years hedged only at time $t = 0$ with the Black-Scholes Delta strategy. Let the underlying be driven by a geometric brownian motion of parameters $\mu = 10\%$ and $\sigma = 40\%$ as in Angelini and Herzel (2009a), for sake of comparison. Moreover, the initial value of the underlying is $S_0 = 100$ and the risk free rate is $r = 0$. In that case, corresponding to $N = 1$, the mean squared hedging error can be computed analytically. Given the parameters of the FFT algorithm, namely $M = 1024$, $\eta = 0.25$, we get the same precision with the Laplace inversion algorithm in Choudhury et al. (1994) setting parameters $A_1 = A_2 = 30$, $l_1 = l_2 = 1$ as in Angelini and Herzel (2009a) and $m = 100$, $n = 135$. For example, the at the money (ATM) relative error between the computed value and the exact one is $1.0 \cdot 10^{-5}$ for both the algorithms. Averaging on the in the money (ITM) strikes, from $K_{ITM} = 25.2979$ to $K_{ATM} = 100$, the absolute value of the relative error is $0.6 \cdot 10^{-6}$, while averaging on the out of the money (OTM) strikes, up to $K_{OTM} = 175.8577$, gives a precision of $0.5 \cdot 10^{-4}$. Considering a vector of strikes going from K_{ITM} to K_{OTM} , the FFT algorithm takes 2.87 seconds, to return results for a vector of 80 strikes while the Laplace algorithm takes 0.77 seconds just for a single strike. All computations were performed in a laptop computer with a 2.0 GHz dual core processor.

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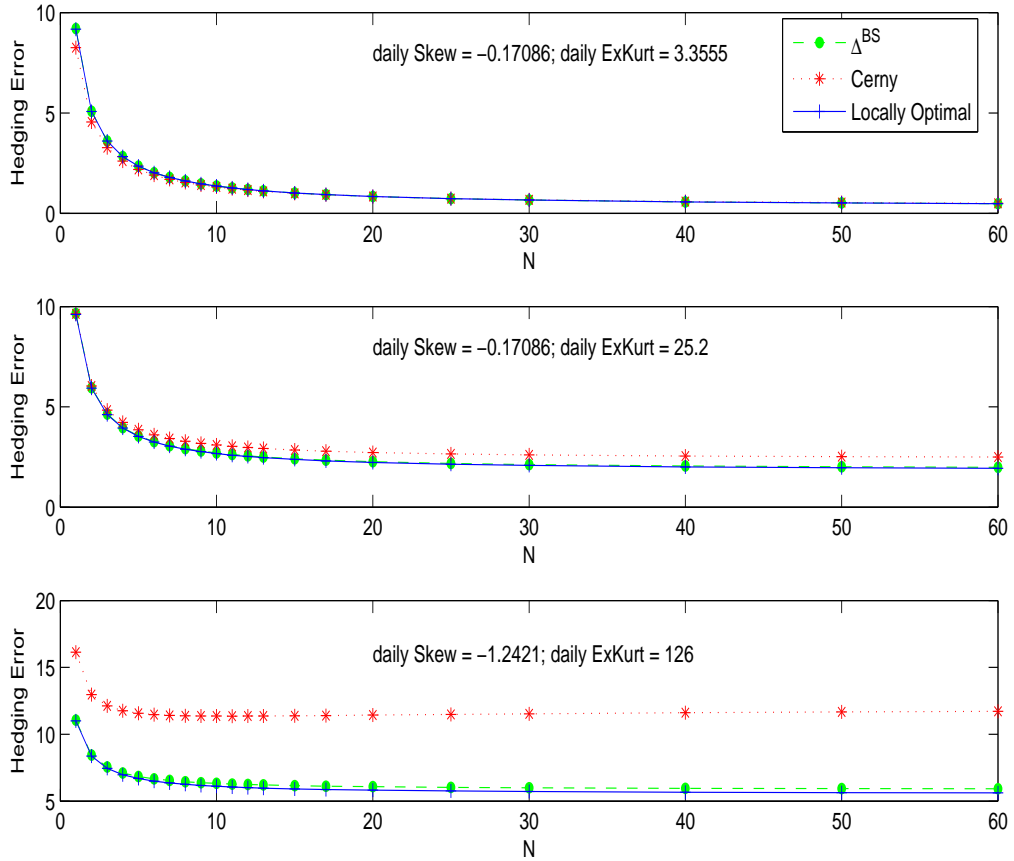


Figure 1 Comparison of Černý's approximating Formula with the semi-explicit Laplace computation as the number of hedging dates increases. It shows the variance of the hedging error when hedging an ATM call of maturity $T = 0.25$ years. The underlying has an initial value of $S_0 = 100$ and it is driven by a NIG process whose skewness and excess of kurtosis (expressed on a daily basis) are different in the three panels. The standard deviation of the process is fixed to about $\tilde{\sigma} = 20\%$ (on a yearly basis), while its expected value is fixed in such a way that the price of the underlying is a martingale. In each panel we plotted Černý's Formula and the Laplace results for the Black-Scholes Delta strategy and for the (locally) optimal strategy.

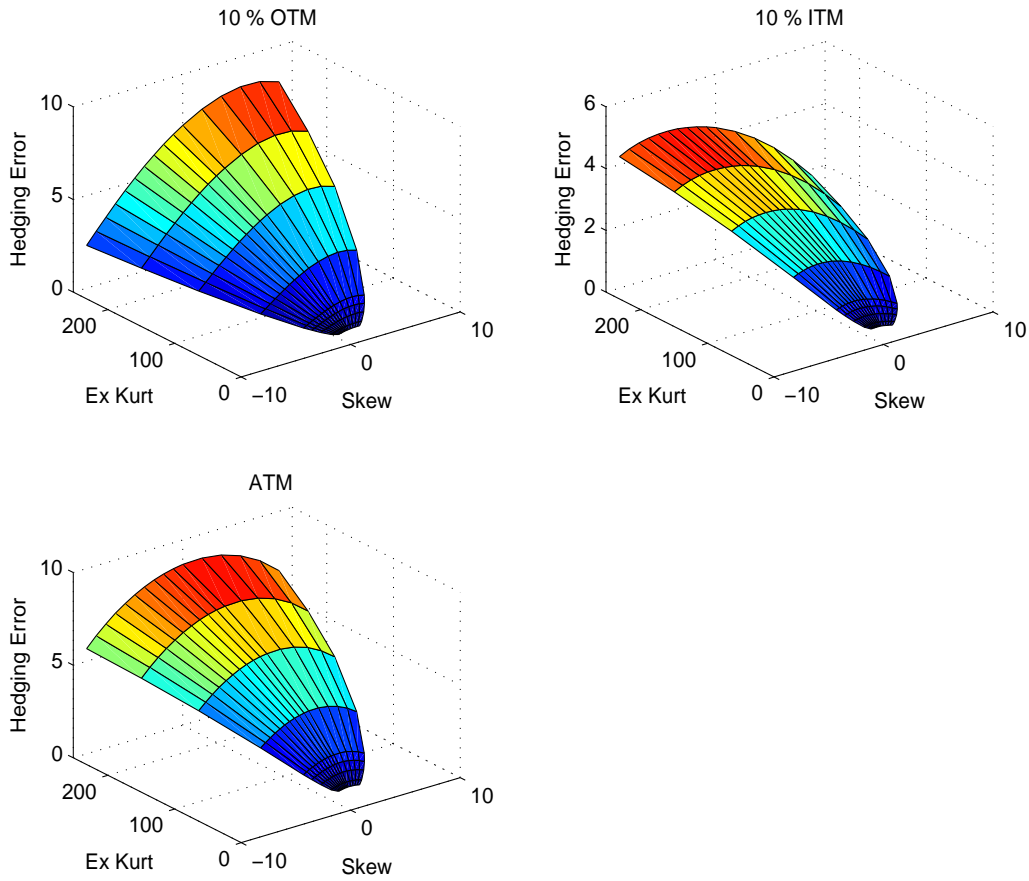


Figure 2 Variance of the hedging error as a function of the skewness and of the excess of kurtosis (expressed on a daily basis) of the underlying log-returns. It is considered the (locally) optimal strategy when hedging a call of maturity $T = 0.25$ years. The underlying has an initial price of $S_0 = 100$. In the upper left (right) figure the call is 10% OTM (ITM), while in the lower figure it is ATM. The underlying log-returns are distributed according to a NIG process. The standard deviation of the distribution is kept constant to $\tilde{\sigma} = 20\%$ (on a yearly basis). The expected value is fixed in such a way that the discounted price is a martingale.

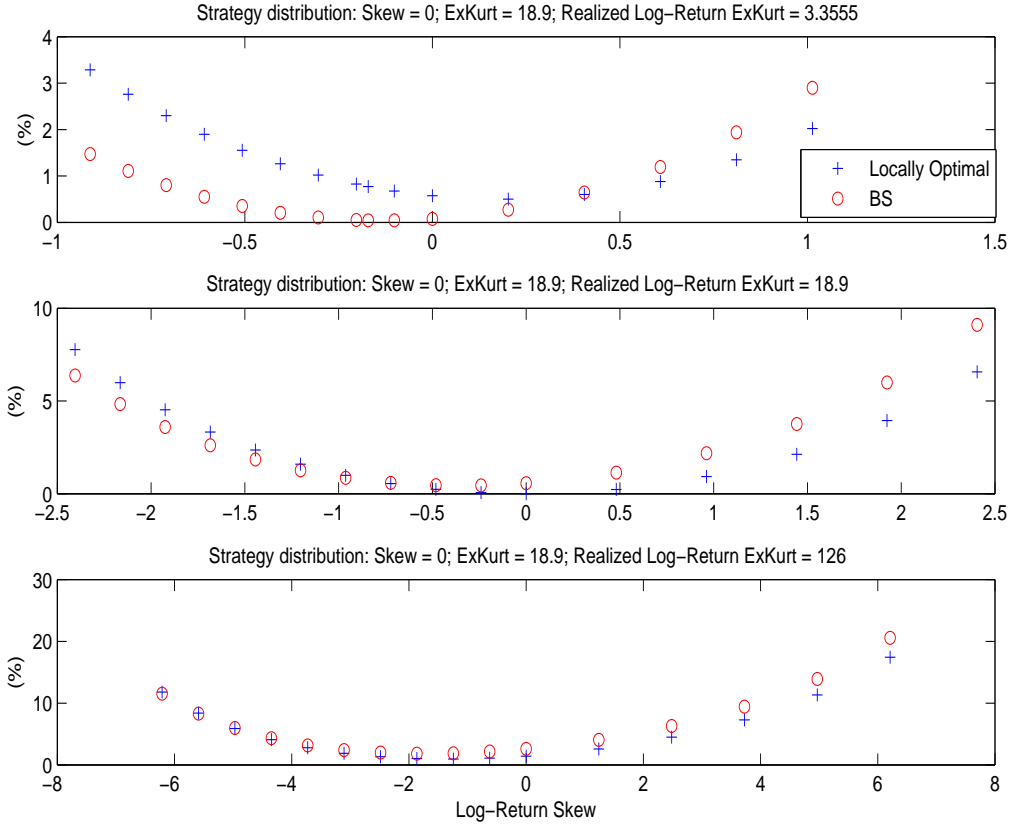


Figure 3 *Skewness and excess of kurtosis misspecification.* On the y-axis is represented $\sigma(\varepsilon)^{mis}/\sigma(\varepsilon)^{LO} - 1$, the percentage increase of the standard deviation of the hedging error of the misspecified strategies with respect to optimal standard deviation computed under the right realized distribution. The results are represented as function of the realized skewness that is reported on the x-axis on a daily basis. The misspecified (locally) optimal strategy has inputs $[Skew, ExKurt] = [0, 18.9]$ (daily). The Black-Scholes Delta strategy is computed at the realized volatility $\tilde{\sigma} = 20\%$ on a yearly basis. Three cases are considered: excess of kurtosis as input of the strategy is over-estimated (top), the same as the realized one (middle), under-estimated (bottom). Results regard an ATM call with maturity $T = 0.25$ years, written on an underlying whose initial value is $S_0 = 100$ that evolves according to the NIG model.

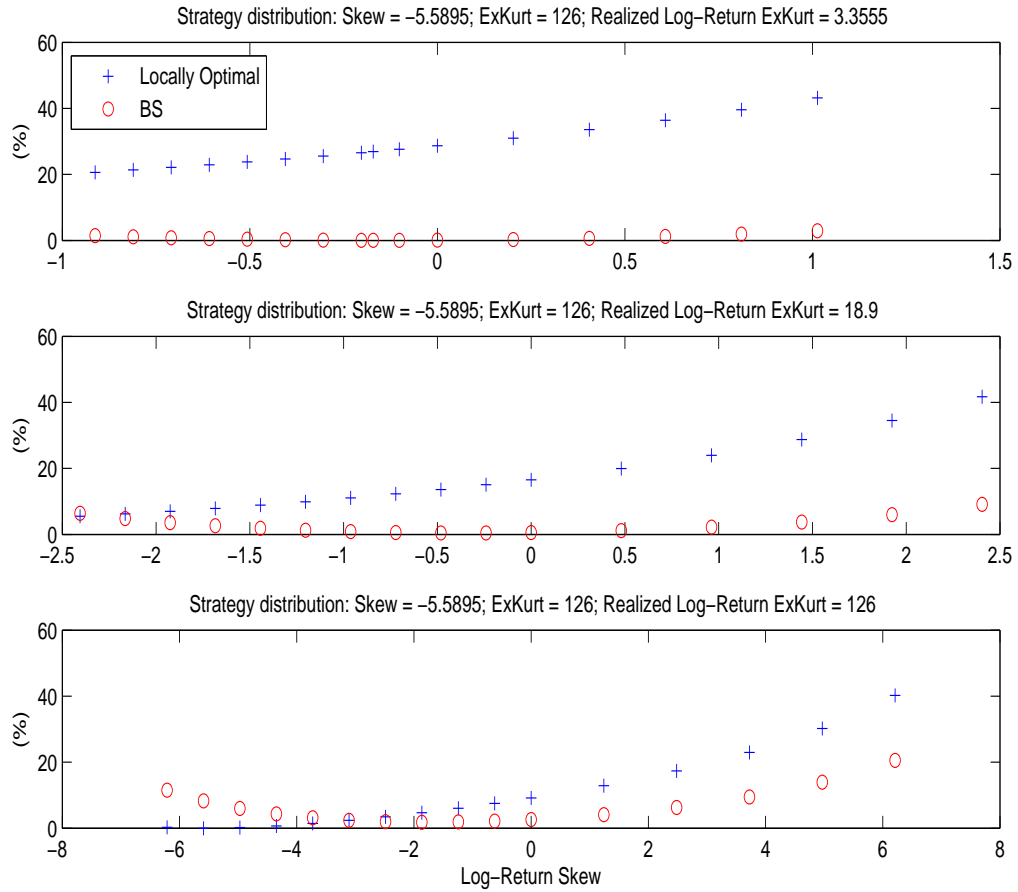


Figure 4 *Skewness and excess of kurtosis misspecification.* On the y-axis is represented $\sigma(\varepsilon)^{mis}/\sigma(\varepsilon)^{LO} - 1$, the percentage increase of the standard deviation of the hedging error of the misspecified strategies with respect to optimal standard deviation computed under the right realized distribution. The results are represented as function of the realized skewness (x-axis). The misspecified (locally) optimal strategy has inputs $[Skew, ExKurt] = [-5.5895, 126]$ (daily).

ExKurt	Skew	ATM		all the strikes	
		average (%)	std dev (%)	average (%)	std dev (%)
18.9000	0	3.6499	4.1171	4.3362	4.6498
3.3555	0	3.9059	4.5073	4.5288	4.8923
3.3555	-0.4054	3.9871	4.9417	4.6006	5.3113
(BS) 0	0	4.0744	4.6741	4.7209	5.0405
3.3555	0.4054	4.1692	4.2901	4.8516	4.7354
18.9000	-1.2027	4.2951	5.5663	4.9942	6.1756
3.3555	-0.8108	4.4297	5.5400	5.0855	5.9268
3.3555	0.8108	4.7658	4.3468	5.5547	4.9012
18.9000	0.9621	4.9702	4.4029	5.9276	5.5273
126.0000	-3.1053	6.7532	5.9647	10.0840	11.6219
18.9000	-2.1648	6.9392	7.5495	7.9560	8.4969
126.0000	0	7.3936	5.1860	11.6716	13.1900
18.9000	1.9243	7.7812	6.1174	9.2062	7.9682
126.0000	2.4842	14.3210	10.5773	19.7210	22.7358
126.0000	-5.5895	16.0546	12.2962	19.8471	19.9195
126.0000	4.9685	25.4476	17.4454	32.2181	32.1305

Table 1: Average and standard deviation over all the distribution in \mathcal{R} of $\sigma(\varepsilon)^{mis}/\sigma(\varepsilon)^{LO} - 1$, the percentage increase in standard deviation of the hedging error of 16 misspecified strategies. BS is the Black-Scholes Delta, the other 15 are the misspecified (locally) optimal strategies according to measures in \mathcal{E} . In the first two columns are indicated the excess of kurtosis and the skewness corresponding to each set of \mathcal{E} , in the second two the results regarding only the ATM option, in the third two the results for all options.